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# On the Structure of a Continuum, Limited and Irreducible Between Two Points.

By Wallace Alvin Wilson.

*Introduction.*

1. The chief result in Z. Janiszewski's thesis * is the theorem to the effect that the necessary and sufficient conditions for a continuum to be the homeomorphic image of a line segment are that it be limited and irreducible between two points and contain no continuum of condensation. Later S. Mazurkiewicz † showed that the last of these conditions is equivalent to requiring that each point of the continuum be a point of the first genre.‡ The peculiarities of irreducible continua not having this last property have led to further study of their characteristics.

In particular, let us consider the simple irreducible continuum defined by the relations: $-1/\pi \leqq x < 0$, $y = \sin^2 1/x$; $x = 0$, $-1 \leqq y \leqq 1$; $0 < x \leqq 1/\pi$, $y = -\sin^2 1/x$. From the standpoint of *connectivity* alone the only difference between this and the ordinary curve which is the graph of $y = \sin x$ is that, when $x = 0$, we have a line segment instead of a point. It appears plausible to infer that, with the proper restrictions, it ought to be possible to resolve any limited irreducible continuum into elements related to one another like the points of a simple arc. That this is true for the special case that the points of the first genre are everywhere dense has been shown elsewhere.§ A solution of the general problem has been given by H. Hahn; ¶ it is the purpose of this paper to give another.

One reason for a second solution is that the process of sub-division can be carried much farther for a very general class of continua than is possible by Hahn's method, and the elements obtained seem more natural. For a

* (A) Z. Janiszewski, "Sur les continus irreductibles entre deux points," *Journ. de l'Ecole Polytechnique*, series II, Vol. 16 (1912).

† (B) S. Mazurkiewicz, "Sur les lignes de Jordan," *Fundamenta Mathematicae*, Vol. I.

‡ That is, a point where the oscillation of the continuum is zero. Another name for this property is *connectedness im kleinen*.

§ See reference (E) under § 2.

¶ (C) H. Hahn, "Über irreduzible Kontinua," *Sitzungsberichte der Akademie der Wissenschaften in Wien*, Vol. 130 (1921).

comparison and examples see §§ 20, 28. Moreover, recent papers on the frontier of a plane region seem to indicate the desirability of as complete a knowledge as possible of the structure of limited irreducible continua.

2. **Some previous results.** The work is based on the notion of the *oscillatory* set of a continuum about a point, which is defined and discussed for continua in general in a previous paper.* In a paper about to be published in the *Transactions of the American Mathematical Society* the following results, among others, were obtained for any limited continuum $ab$, irreducible between $a$ and $b$.

($a$) If $A$ and $B$ are real sub-continua (in the generalized sense †) of $ab$, containing $a$ and $b$ respectively, and if $A \cdot B = 0$, then $\overline{ab - (A + B)}$ is a continuum joining $A$ and $B$ and is the only such sub-continuum.

($b$) If $x$ is any point of $ab$ and if $V_\delta(x)$ denotes those points of $ab$ whose distance from $x$ is less than $\delta$, there is a unique sub-continuum $X_\delta$ of $ab$ which is irreducible about ‡ $V_\delta(x)$. Let the divisor of the sets $X_\delta$ for $\delta > 0$ be denoted by $X$. Then $X$ is a continuum or a point, and is the *oscillatory set* § of $ab$ about $x$.

($c$) If $X$ is the oscillatory set of $ab$ about $x$, the diameter of $X$ is the oscillation ¶ $\sigma(x)$ of $ab$ about $x$. If $\sigma(x) = 0$, $X = x$. If $X \neq x$, the oscillation of $ab$ about each point of $X$ is greater than zero.

($d$) If $X_a$ denotes the saturated semi-continuum of $ab - X$ containing $a$, and $X_b$ has a similar meaning for $b$, then

$$ab = \bar{X}_a + X + \bar{X}_b, \quad \bar{X}_a \cdot X \neq 0, \quad X \cdot \bar{X}_b \neq 0;$$

and

$$X_a \cdot X_b = \bar{X}_a \cdot X_b = X_a \cdot \bar{X}_b = 0.$$

($e$) If neither $X_a$ nor $X_b$ is void, and $x$ lies in both $\bar{X}_a$ and $\bar{X}_b$, we call $X$ complete. If the oscillatory set $A$ of $ab$ about $a$ does not contain $b$ and $\bar{A}_b$

* (D) W. A. Wilson, "On the Oscillation of a Continuum," *Transactions of the American Mathematical Society*, Vol. 27, No. 4, pp. 429-440.

† That is, $A$ and $B$ may contain only the points $a$ and $b$.

‡ That is, irreducible with respect to the property of containing all the points of $V_\delta(x)$.

§ (E) W. A. Wilson, "Some Properties of a Continuum, limited and irreducible between two Points," *Transactions of the American Mathematical Society*, Vol. 28. This definition is equivalent to that given in reference (D). It is also equivalent to a more general form, in which $V_\delta(x)$ is any region relative to $ab$. This last form makes it easy to show that the oscillatory set is an invariant of Analysis Situs.

¶ For the meaning of this see reference (B).

contains $a$, we call $A$ complete. Under like conditions $B$ is complete. Also if $X$ is identical with $A$ (or $B$) and $A$ (or $B$) is complete, $X$ is complete. In the example of § 1 the oscillatory set about the origin is the segment joining the points $(0, -1)$ and $0, 1)$ and it is complete; the oscillatory set about any other point of this segment is not complete.

For complete oscillatory sets we have the following properties:

$ab = X_a + X + X_b, \quad X_a \cdot X = X \cdot X_b = X_a \cdot X_b = 0;$
$X_a + X$ and $X + X_b$ are continua;
if $X = x$, it is complete;
$X$ is a continuum of condensation or a point;
if $y$ is a point of $X$, $Y$ is a part of $X$;
if $y$ is a point of $X$ and also of $\bar{X}_a$ and $\bar{X}_b$, $Y = X$;
if $y$ is a point of $X_a$ (or $X_b$), the oscillatory set of $ab$ about $y$ is a part of $X_a$ (or $X_b$).
$ab = A + A_b = B_a + B, \quad A \cdot A_b = B_a \cdot B = 0.$

3. **Notation.** The ordinary notation of the theory of aggregates is employed, with the following modifications.

If $A$ and $B$ are aggregates, the notation $A < B$ means that $A$ is a real part of $B$. If $A$ is a part of $B$ and may be identical with $B$, we write $A \leq B$.

If $A$ is the common part of the system of aggregates $\{C\}$, we write $A = Dv[C]$. If $A$ contains every element of each of the aggregates $\{C\}$, we write $A = U[C]$. If also no two of the sets $\{C\}$ have common elements, we write $A = \Sigma[C]$.

The phrase "irreducible continuum $ab$" means that $ab$ is irreducible between $a$ and $b$.

The symbol $V_\delta(x)$ is defined in § 2(*b*). Likewise, if $A \leq C$, $V_\delta(A)$ means the set of points of $C$ whose distance from $A$ is less than $\delta$.

If $x$ is a point of $ab$, the symbols $X_\delta$, $X$, $X_a$, and $X_b$ always have the meanings defined in § 2 (*b*) and (*d*), unless the contrary is expressly stated.

4. **Summary of principal results.** The relation $ab = X_a + X + X_b$ given in § 2 (*e*) indicates the possibility of dividing $ab$ into complete oscillatory sets without common points. The principal theorems may be summarized as follows.

THEOREM I. If $ab$ is a limited irreducible continuum, the necessary and sufficient condition that it can be sub-divided into complete oscillatory sets without common points is that $ab$ contain no indecomposible continuum which is not a continuum of condensation.

THEOREM II. If $K = \{X\}$ denotes the aggregate of complete oscillatory sets of a continuum satisfying the condition of Theorem I, there is a similar correspondence between $K$ and a unit segment $T = (0 \leqq t \leqq 1)$. If this correspondence is denoted by $X = f(t)$, then $f(t)$ is supra-continuous.*

A number of minor results accompany these theorems.

*Some General Theorems.*

5. In order to obtain the results indicated in § 4 we need certain general theorems not derived in either of the papers referred to above. The following sections (§§ 5-10) will be devoted to these.

THEOREM. *Let $ab$ be a limited irreducible continuum and let $x$ be one of its points differing from $a$ and $b$. Let $A$ and $B$ be any real sub-continua of $ab$ containing $a$ and $b$ respectively, but not $x$. Let $L$ be the sub-continuum of $ab$ irreducible between $A$ and $B$. Then, if $X$ denotes the oscillatory set of $ab$ about $x$, $X = Dv[L]$ for all possible sub-continua $A$ and $B$.*

PROOF. For a given $A$ and $B$ and $\delta > 0$ sufficiently small, $A \cdot V_\delta(x) = B \cdot V_\delta(x) = 0$. Hence $V_\delta(x) < L$ and, by § 2 $(b)$, $X \leq L$. This gives

$$X \leq Dv[L]. \tag{1}$$

For any $\delta > 0$ there exist by a theorem of Janiszewski † sub-continua $A$ and $B$ containing points of $V_\delta(x)$. But then $A \cdot X_\delta \neq 0$ and $B \cdot X_\delta \neq 0$. Then $L \leq X_\delta$ by § 2 $(a)$, and hence $Dv[L] \leq X_\delta$ for every $\delta > 0$. This gives

$$Dv[L] \leq X. \tag{2}$$

Relations (1) and (2) give the theorem.

The statement and proof of the analogous theorem for the case that $x = a$ or $x = b$ are obvious.

6. THEOREM. *No two complete oscillatory sets of a limited irreducible continuum $ab$ have common points unless they are identical.*

PROOF. I. Let the oscillatory sets $X$ and $Y$ of $ab$ about $x$ and $y$, respectively, be complete and let neither be identical with $A$ or $B$. Then by § 2 $(e)$.

$$ab = X_a + X + X_b, \quad X_a \neq 0, \quad X_b \neq 0.$$

If $y$ is a point of $X$, $Y = X$ by § 2 $(e)$. If $y$ is a point of $X_a$, then $Y < X_a$

* For the meaning of this term see § 26.

† See reference (A), p. 100.

by § 2 (*e*). If $y$ is a point of $X_b$ then $Y < X_b$. Hence either $Y = X$ or $Y \cdot X = 0$.

II. Let $X = A$. Then $ab = X + X_b$. Then, as above, either $Y = X$ or $Y < X_b$, in which case $Y \cdot X = 0$. The case that $X = B$ is treated in like manner.

7. THEOREM. *Let $ab$ be a limited irreducible continuum, let $X$ be the oscillatory set about $x$, and let $X_a \neq 0$. Then $\bar{X}_a$ is irreducible between $a$ and every point of $X \cdot \bar{X}_a$ and is the only sub-continuum of $ab$ irreducible between these points.*

PROOF. Let $y$ be a point of $X \cdot \bar{X}_a$ and let $ay$ be a sub-continuum of $ab$ irreducible between $a$ and $y$. As $X + \bar{X}_b$ is a continuum by § 2 (*d*), we have

$$ab = ay + X + \bar{X}_b.$$

But $X_a \cdot (X + \bar{X}_b) = 0$ by § 2 (*d*); hence $X_a < ay$. Then $\bar{X}_a \leq ay$. As $\bar{X}_a$ contains both $a$ and $y$, this gives $ay = \bar{X}_a$, since otherwise $ay$ would not be irreducible.

COROLLARY 1. *Let $ab$ be a limited irreducible continuum, let $X$ be the oscillatory set about $x$, let $X_a \neq 0$, and let $x$ be a point of $\bar{X}_a$.* Then $X_a + X$ *is a continuum and* $\bar{X}_a \leq X_a + X$.

PROOF. Let $X_\delta$ be the sub-continuum of $ab$ irreducible about $V_\delta(x)$, $\delta > 0$. Then it is evident that $X_a + X_\delta$ contains a continuum joining $a$ and $x$, and, in consequence of the above theorem, $\bar{X}_a \leq X_a + X_\delta$ for all $\delta > 0$. Hence

$$\bar{X}_a \leq Dv[X_a + X_\delta] = X_a + X.$$

Since $X \cdot \bar{X}_a \neq 0$, $X_a + X$ is connected. Also $\overline{X_a + X} = \bar{X}_a + X \leq X_a + X$. Hence $X_a + X$ is closed, as well as connected.

COROLLARY 2. *Let $ab$ be a limited irreducible continuum and let the oscillatory set $B$ of $ab$ about $b$ be complete. Then $ab$ is irreducible between $a$ and every point of $B$.*

For $\bar{B}_a = ab$ and $B = B \cdot \bar{B}_a$.

8. It is easy to see from examples that, if $C$ and $C'$ are continua and $C < C'$, the oscillatory set of $C$ about one of its points $x$ is not necessarily a part of that of $C'$ about $x$. We do, however, have the following theorem.

THEOREM. *Let $ab$ be a limited irreducible continuum and let $x$ be a point of $ab$. Let $ax$ be a sub-continuum of $ab$ irreducible between $a$ and $x$.*

*Then the oscillatory set of $ax$ about $x$ is a part of that of $ab$ about $x$, and* $\sigma_{ax}(x) \leqq \sigma_{ab}(x)$.

PROOF. Let $A$ be any sub-continuum of $ax$ containing $a$ but not $x$, and let $B$ be likewise defined for any sub-continuum $xb$ of $ab$ irreducible between $x$ and $b$. Now $A \cdot B = 0$, since $ab$ is irreducible and $x$ lies in neither $A$ nor $B$. Let $K$ be the sub-continuum of $ab$ irreducible between $A$ and $B$. Then

$$ab = A + K + B, \quad A \cdot K \neq 0, \quad B \cdot K \neq 0.$$

If $B \cdot ax = 0$, $\quad ax \leq A + K$.

If $B \cdot ax \neq 0$, $\quad ab = ax + B$. Then $K \leq ax$, since $K$ is the only sub-continuum of $ab$ irreducible between $A$ and $B$. As $A < ax$, this gives $A + K \leq ax$. But $x$ is a point of $K$ and $ax$ is irreducible; hence $A + K = ax$. Thus in both cases we have

$$ax \leq A + K.$$

Let $S = K \cdot ax$. Then $S$ is closed and $x$ is a point of $S$, since it is not a point of $A$. Now there is a partition of $S$ into two closed sets $S_1$ and $S_2$ without common points such that neither set contains both $x$ and a point of $A$, or there is not. In the former case let $x$ be a point of $S_2$. Then

$$ax = A \cdot ax + K \cdot ax = A + S_1 + S_2.$$

The sets $A + S_1$ and $S_2$ are closed, and $(A + S_1) \cdot S_2 = A \cdot S_2 + S_1 \cdot S_2 = 0$, since $A \cdot S_2 = 0$. This is false, since $ax$ is a continuum.

Hence the latter case holds and $S$ contains a sub-continuum $S'$ irreducible between $x$ and $A$.* Then $A + S'$ is a continuum containing $a$ and $x$, and, since $A + S' \leq ax$, we have

$$A + S' = ax.$$

Let $U_\delta(x)$ denote those points of $ax$ whose distance from $x$ is less than $\delta$, $\delta > 0$. For $\delta$ sufficiently small $A \cdot U_\delta(x) = 0$ and $U_\delta(x) < S'$. Hence if $D$ is the oscillatory set of $ax$ about $x$, $D \leq S'$ and consequently

$$D \leq K.$$

But $A$ and $B$ were *any* sub-continua of $ax$ and $xb$, respectively, containing $a$ and $b$, but not $x$. From § 5 we see that for any $\epsilon > 0$, $K \leq V_\epsilon(X)$ if only

* This follows from an easily generalized theorem of Janiszewski. See reference (A), p. 109.

$Dist(x, A)$ and $Dist(x, B)$ are less than some positive $\eta$. Obviously $A$ and $B$ can be so chosen. Hence for any $\epsilon > 0$, $D \leq V_\epsilon(X)$, which in turn gives

$$D \leq X.$$

By § 2 (*c*), $\sigma_{ax}(x)$ and $\sigma_{ab}(x)$ are the diameters of $D$ and $X$, respectively. Then

$$\sigma_{ax}(x) \leq \sigma_{ab}(x).$$

COROLLARY. *Let $ax$ and $xb$ be limited irreducible continua and $E$ and $F$ be the oscillatory sets of $ax$ and $xb$, respectively, about $x$. Let $ab = ax + xb$ and $ab$ be irreducible between $a$ and $b$. Let $X$ be the oscillatory set of $ab$ about $x$. Then $X = E + F$.*

For, by the above theorem, $E + F \leq X$. But from the definition of oscillatory set it is obvious that $X \leq E + F$. Hence $X = E + F$.

9. THEOREM. *Let $ab$ be a limited irreducible continuum and let $X$ be the oscillatory set about the point $x$. Let $X_a \neq 0$, let $x$ be a point of $\bar{X}_a$, and let $E$ be the oscillatory set of $\bar{X}_a$ about $x$. Then $E = X \cdot \bar{X}_a$. If $E_a$ denotes the saturated semi-continuum of $\bar{X}_a - E$ containing $a$, $E_a = X_a$.*

PROOF. $\bar{X}_a$ is an irreducible $ax$ by § 7. Then by § 8, $E \leq X$ and hence

$$E \leq X . \bar{X}_a. \tag{1}$$

Let $U_\delta(x)$ denote those points of $\bar{X}_a$ whose distance from $x$ is less than $\delta$. Since $x$ is a point of $\bar{X}_a$, $X_a \cdot U_\delta(x) \neq 0$. Let $E_\delta$ denote the sub-continuum of $\bar{X}_a$ irreducible about $U_\delta(x)$. Then by § 7, $\bar{X}_a \leq X_a + E_\delta$ for every $\delta > 0$; hence

$$\bar{X}_a \leq X_a + E. \tag{2}$$

From § 7, Corollary 1, it is easily deduced that $\bar{X}_a - X_a = X \cdot \bar{X}_a$; consequently relation (2) gives $E \geq X \cdot \bar{X}_a$. This with relation (1) gives $E = X \cdot \bar{X}_a$.

Then $\bar{X}_a - E = \bar{X}_a - X \cdot \bar{X}_a = X_a$. Since $\bar{X}_a - E$ is a semi-continuum, we have $E_a = X_a$.

COROLLARY. *Under the hypotheses of the above theorem, $E$ is the oscillatory set of $\bar{X}_a$ about any point of $E$.*

For $E$ is complete and, if $y$ is a point of $E$, it is a point of $\bar{E}_a$. But then by § 2 (*e*) the oscillatory set of $\bar{X}_a = ax$ about $y$ is the same as $E$, that about $x$.

REMARK. The set $E$ defined in the above theorem may be regarded as a kind of left-hand oscillatory set about $x$. If $\bar{X}_b \neq 0$ and contains $x$, we would have in like manner a set which might be regarded as a right-hand oscillatory set. As the next theorem shows, the two together would make up the (complete) oscillatory set of $ab$ about $x$.

10. THEOREM. *Let $ab$ be a limited irreducible continuum. Let the oscillatory set $X$ of $ab$ about $x$ contain neither $a$ nor $b$ and be complete. Let $E$ be the oscillatory set of $\bar{X}_a$ about $x$, and let $F$ be that of $\bar{X}_b$ about $x$. Then $(a)$ $X = E + F$; $(b)$ $E$ is the oscillatory set of $ab$ about each point of $X - F$; $(c)$ $F$ is the oscillatory set of $ab$ about each point of $X - E$; and $(d)$ $X$ is the oscillatory set of $ab$ about each point of $E \cdot F$.*

PROOF. Since $X$ is complete, $ab = \bar{X}_a + \bar{X}_b$ Then $X = X \cdot \bar{X}_a + X \cdot \bar{X}_b = E + F$ by § 9. This gives $(a)$.

To prove $(c)$ note that from the proof of § 9 we have $\bar{X}_a = X_a + E$. Hence, if $y$ is a point of $X - E$, it is not a point of $\bar{X}_a$. Hence, if $\delta$ is small enough, $V_\delta(y) < \bar{X}_b$ and consequently the oscillatory set of $ab$ about $y$ coincides with that of $\bar{X}_b$ about $y$, namely with $F$, by the corollary to § 9. A similar proof takes care of $(b)$.

Statement $(d)$ is proved elsewhere, as stated in § 2 $(e)$.

*Necessary and sufficient conditions for the decomposition of a limited continuum, irreducible between two points, into complete oscillatory sets.*

11. The above question amounts to determining the necessary and sufficient conditions for each point of such a continuum to lie on one and only one complete oscillatory set. That this is not true for every such continuum follows from the existence of indecomposible continua. For, if $ab$ is indecomposible and $x$ is one of its points, $ab$ is irreducible between $x$ and a point in every $V_\delta(x)$.* Hence every $X_\delta = ab$ and consequently $X = ab$. Then $ab - X = 0$ and $X$ is not complete.

We now proceed to investigate the conditions under which the above decomposition of $ab$ is possible. In general four possibilities must be considered in relation to a given point $x$ and its oscillatory set $X$:

$(a)$ $x$ is a point of both $\bar{X}_a$ and $\bar{X}_b$;

$(b)$ $x$ is a point of neither $\bar{X}_a$ nor $\bar{X}_b$;

* (F) S. Mazurkiewicz, "Une théorème sur les continus indécomposables," *Fundamenta Mathematicae*, Vol. I (1920), p. 38.

(*c*) $x$ is a point of $\bar{X}_a$, but not of $\bar{X}_b$;
(*d*) $x$ is a point of $\bar{X}_b$, but not of $\bar{X}_a$.

Before considering the general case let us consider the case that $x = a$ or $x = b$. If $x = b$, we have two possibilities only:

(*a*) $b$ is a point of $\bar{B}_a$; (*b*) $b$ is not a point of $\bar{B}_a$.

In the first case $B$ is complete; the second is discussed in the next section.

12. THEOREM. *Let $ab$ be a limited irreducible continuum and let $B$ be the oscillatory set about $b$. If $B$ is not complete, it is an indecomposible continuum which is not a continuum of condensation.*

PROOF. By the definition of complete oscillatory sets, either $B = ab$ or $\bar{B}_a \neq 0$ and does not contain $b$.

If in the former case $B$ were decomposible, we should have a contradiction. For then $B = ab = B_1 + B_2$, where $B_1$ and $B_2$ are continua, $B_1 \neq B_2 \neq B$, and neither $B_1$ nor $B_2$ contains both $a$ and $b$. Let $B_1$ contain $a$ and $B_2$ contain $b$. For a positive $\delta$ sufficiently small $B_1 \cdot V_\delta(b) = 0$ and $V_\delta(b) < B_2$. As $B_2$ is a sub-continuum of $ab$ containing $V_\delta(b)$, we have $B \leq B_2$, which is false.

In the latter case $B_a \neq 0$ and we know from § 2 (*d*) that $ab = \bar{B}_a + B$. Now $B$ is irreducible between $\bar{B}_a$ and $b$. For otherwise $ab = \bar{B}_a + B'$, where $B'$ is a continuum containing $b$, and $B' < B$. Since $b$ is not a point of $\bar{B}_a$, some $V_\delta(b) < B'$. Then $B \leq B'$, which is a contradiction.

Let $c$ be a point of $B \cdot \bar{B}_a$. If $B$ were not indecomposible, $B$ would equal $B_1 + B_2$, where $B_1$ and $B_2$ are continua and $B_1 \neq B_2 \neq B$. The previous paragraph shows that neither contains both $b$ and $c$; let $B_1$ contain $c$ and $B_2$ contain $b$. Then some $V_\delta(b)$ contains no point of $\bar{B}_a + B_1$; i. e., $V_\delta(b) < B_2$. Then $B \leq B_2$; which is a contradiction. Hence in this case also $B$ is indecomposible. That it is not a continuum of condensation follows from the fact that $b$ is not a point of $\bar{B}_a$.

COROLLARY. *Under the hypotheses of the above theorem, $B$ is the oscillatory set of $ab$ about every point of $B$ not in $\bar{B}_a$.*

For, if $c$ is such a point and $\delta$ is small enough, $\bar{B}_a \cdot V_\delta(c) = 0$. Hence $V_\delta(c) < B$ and the oscillatory set about $c$ is contained in $B$. It cannot be less than $B$ since $B$ is indecomposible and hence irreducible between $c$ and a point in every $V_\delta(c)$.

It is evident from § 2 (*e*) and the above theorem that there are two possible alternatives with regard to the oscillatory sets of the points $a$ and $b$ of the limited irreducible continuum $ab$. (*a*) The oscillatory set may be complete. If so, it is a continuum of condensation and is the oscillatory set of each of its points. It may even be indecomposible. (*b*) If it is not complete, it is an indecomposible continuum which is not a continuum of condensation, and it is the oscillatory set of each of its relatively inner points.

The above theorem may also be stated as follows. *If $ab$ contains no indecomposible continuum which is not a continuum of condensation, then the oscillatory sets about $a$ and $b$ are complete.* With this theorem as a basis we now proceed to treat the four alternatives in the general case.

13. THEOREM. *Let $ab$ be a limited irreducible continuum and $x$ be a point differing from $a$ and $b$. Let $X$ be the oscillatory set about $x$ and let $x$ lie in neither $\bar{X}_a$ nor $\bar{X}_b$. Then $X$ is indecomposible or is the union of two indecomposible continua, and none of these is a continuum of condensation of $ab$.*

PROOF. We assume that $X$ contains neither $a$ nor $b$; the proof will hold if suitably modified when $X_a$ or $X_b$ or both are void.

I. If $X$ is indecomposible, we need only to show that it is not a continuum of condensation. Now $ab = \bar{X}_a + X + \bar{X}_b$, whence $ab - X \leq \bar{X}_a + \bar{X}_b$. Then if we had $ab = \overline{ab - X}$, $ab$ would equal $\bar{X}_a + \bar{X}_b$, which is impossible as neither $\bar{X}_a$ nor $\bar{X}_b$ contains $x$.

II. Let $X$ be decomposible. We first note that for *any* decomposition of $X$ into two continua $X_1$ and $X_2$ such that $X_1 \neq X_2 \neq X$, $x$ is a point of $X_1 \cdot X_2$. For $ab = \bar{X}_a + X_1 + X_2 + \bar{X}_b$. If $x$ were not a point of $X_2$, there would be a $\delta > 0$ so small that $V_\delta(x) < X_1$. This would make $X \leq X_1$, which is false. In like manner we can show that $x$ is a point of $X_1$.

It may be shown by the usual methods * employed in proving the existence of irreducible aggregates of various kinds that there is a decomposition of $X$ such that, if $X'$ is a real sub-continuum of $X_1$, then $X' + X_2 < X$, and, if $X''$ is a real sub-continuum of $X_2$, then $X_1 + X'' < X$. Let then $X = X_1 + X_2$ be such an "irreducible decomposition."

Since $X \cdot \bar{X}_a \neq 0$, either $\bar{X}_a \cdot X_1 \neq 0$ or $\bar{X}_a \cdot X_2 \neq 0$. Let $\bar{X}_a \cdot X_1 \neq 0$.

* See reference (D), § 6.

Then $\bar{X}_b \cdot X_1 = 0$. For, if not, $ab = \bar{X}_a + X_1 + \bar{X}_b$. But then for some positive $\delta$, $V_\delta(x) < X_1$ and consequently $X \leq X_1$, which is false. Since $\bar{X}_b \cdot X_1 = 0$, $\bar{X}_b \cdot X_2 \neq 0$. Likewise $\bar{X}_a \cdot X_2 = 0$. Thus we have

$$ab = \bar{X}_a + X_1 + X_2 + \bar{X}_b, \tag{1}$$

where each continuum has points in common with only the ones immediately preceding or following.

From § 2 (*a*) we learn that there is a unique sub-continuum $D$ of $ab$ irreducible between $a$ and $X_2 + \bar{X}_b$. Then from relation (1) we have

$$D \leq \bar{X}_a + X_1 \quad \text{and} \quad \bar{X}_a < D, \tag{2}$$

since $\bar{X}_a \cdot (X_2 + \bar{X}_b) = 0$ and $ab = D + X_2 + \bar{X}_b$. Let $D'$ be the sub-continuum of $ab$ irreducible between $\bar{X}_a$ and $X_2 + \bar{X}_b$. Then $D' \leq X_1$ and $D' \cdot X_2 \neq 0$, as relation (1) shows. Since $D' + X_2 \leq X$ and $X$ is irreducible between $\bar{X}_a$ and $\bar{X}_b$, $D' + X_2 = X$. Then $D' = X_1$, for $X = X_1 + X_2$ was an irreducible decomposition. But $D$ also joins $\bar{X}_a$ and $X_2 + \bar{X}_b$; hence $X_1 = D' \leqq D$. Therefore $\bar{X}_a + X_1 \leq D$; whence relation (2) gives $D = \bar{X}_a + X_1$.

Since $D = \bar{X}_a + X_1$ is irreducible between $a$ and $X_2 + \bar{X}_b$, it is a unique irreducible $ax$, for $x$ is a point of $X_2 + \bar{X}_b$. In precisely the same fashion we find that $X_2 + \bar{X}_b$ is a unique irreducible $xb$.

Now let $E$ be the oscillatory set of $ax$ about $x$ and let $F$ be that of $xb$ about $x$. By § 8, Corollary,

$$X = E + F.$$

But $E < ax = \bar{X}_a + X_1$ and $X = X_1 + X_2$. Hence by relation (1) $E \leq (\bar{X}_a + X_1)(X_1 + X_2) \leq X_1$. Likewise $F \leq X_2$. But the partition $X = X_1 + X_2$ was irreducible, whence $E = X_1$ and $F = X_2$.

Since $x$ is not a point of $\bar{X}_a$, we can apply § 12 and this shows that $X_1$ is indecomposible. Nor is $X_1$ a continuum of condensation. For $ab - X_1 \leqq \bar{X}_a + X_2 + \bar{X}_b$. Hence $\overline{ab - X_1} \leq \bar{X}_a + X_2 + \bar{X}_b < ab$, as seen in relation (1). In like manner $X_2$ is indecomposible and is not a continuum of condensation.

14. Either of the two possibilities discussed in the theorem of § 13 may be realized in practice. Consider the example of an indecomposible continuum given in Janiszewski's thesis.* This is composed of an infinity

* See reference (A), p. 114.

of straight line segments, of which the first three segments $cu + uv + vw$ are sides of a square and the remainder lies in the square, together with its limiting points $\{d\}$. In the paper referred to it is shown that this is irreducible between $c$ and any such limiting point $d$ and is indecomposible; let us denote it by $cd$. Let $cb$ be a segment in the plane of $cd$ meeting $cd$ only in $c$ and let $ad$ be a segment perpendicular to this plane. Obviously $ab = ad + cd + cb$ is irreducible between $a$ and $b$. If $x$ is any point of $cd$, the oscillatory set of $ab$ about $x$ is $cd$. Thus $X$ is indecomposible; $X_a = ad - d$; and $X_b = cb - c$.

If now $cd$ is reflected about the line $cu$, we have two indecomposible continua, $cd$ and $cd'$, having in common the segment $cu$. If $ad$ and $d'b$ are segments perpendicular to the common plane of these continua, $ab = ad + cd + cd' + d'b$ is irreducible between $a$ and $b$. Let $x$ lie on $cu$. Then $X = cd + cd'$, $X_a = ad - d$, and $X_b = d'b - d'$. Here $X$ is the union of two indecomposible continua.

We evidently need to consider only one of the mixed cases (Cases $(c)$ and $(d)$ in § 11), say the one where $x$ is a point of $\bar{X}_b$ but not of $\bar{X}_a$. This may happen in two obvious ways. First, let $ab$ be the irreducible continuum defined in § 1. Let $z$ be a point lying on the $y$-axis between the origin and the point $(0, -1)$; then $Z$ is the segment joining the points $(0, 0)$ and $(0, -1)$. Evidently $z$ is a point of $\bar{Z}_b$ but not a point of $\bar{Z}_a$. Hence $Z$ is not complete. But the oscillatory set about the origin is the segment joining $(0, -1)$ and $(0, 1)$ and it is complete. Moreover it contains $Z$ as a part.

A second possibility is the following. Let $cd$ be the indecomposible continuum used above, let $ad$ be a segment perpendicular to the plane of $cd$, and let $ce$ denote the segment $cu$ prolonged to twice the length of $cu$. Let $b$ be a point in the plane of $cd$, not on $ce$ or in the square containing $cd$, and let $cb$ denote a wavy line starting at $b$, which oscillates indefinitely and has the segment $ce$ as a limiting line, but does not cut $cd + ce$. Then $ab = ad + cd + cb$ is irreducible. If $x$ is a point on $ce$, it is evident that $X = cd + ce$, $X_a = ad - d$, and $X_b = cb$. Here $X$ is not complete and is composed of the indecomposible continuum $cd$ and the continuum of condensation $ce$. Moreover $X$ is a part of no complete oscillatory set.

A less obvious third possibility is the following. Let $ab$ consist of a circumference $C$ of center $c$, a line $ae$ joining a point $e$ on $C$ to a point $a$ without $C$, a spiral starting at $c$ and converging to $C$, and a line $cb$ perpendicular to the plane of these figures. Let $x$ be a point on $C$ different from $e$. Then $X = C$, $X_a = ae - e$, $X_b = ab - (ae + C)$. Here there is no indecomposible

continuum involved, but $x$ is not a point of $\bar{X}_a$, for $\bar{X}_a = ae$. Note that there are here two irreducible sub-continua $ax$ of $ab$. This fact suggests the following lemma.

15. LEMMA. *Let $x$ be a point of the limited irreducible continuum $ab$. Let $X$ be the oscillatory set about $x$, let $X_a \neq 0$, and let $X_b \neq 0$. Let $x$ be a point of $\bar{X}_b$, but not of $\bar{X}_a$. Then there is a point $x'$ on $X \cdot \bar{X}_b$ such that there is a unique irreducible sub-continuum $ax'$ of $ab$ and $\bar{X}_b$ is a unique $x'b$.*

PROOF. Let $F = X \cdot \bar{X}_b$. By § 7 $\bar{X}_b$ is irreducible between $b$ and every point of $F$. Let $D$ be the sub-continuum of $ab$ irreducible between $a$ and $\bar{X}_b$. Since $\bar{X}_a + X$ is a continuum and $a + F < \bar{X}_a + X$, $D \leq \bar{X}_a + X$. Then $D \cdot X_b = 0$. Since $\bar{X}_b \leq X + X_b$, by § 7, Corollary 1, $D \cdot \bar{X}_b \leq D \cdot X$ or $D \cdot \bar{X}_b \cdot X \neq 0$. Hence $D \cdot F$ contains at least one point $x'$ and $D$ is a unique irreducible $ax'$, while $\bar{X}_b$ is a unique irreducible $x'b$.

16. THEOREM. *Let $x$ be a point of the limited irreducible continuum $ab$. Let the oscillatory set $X$ contain neither $a$ nor $b$. Let $x$ be a point of $\bar{X}_b$, but not of $\bar{X}_a$. Let $ax'$ be the unique sub-continuum of $ab$ irreducible between $a$ and $X \cdot \bar{X}_b$. Let the oscillatory set $E$ of $ax'$ about $x'$ be not complete. Then $E$ is indecomposible and not a continuum of condensation.*

PROOF. That $E$ is indecomposible was proved in § 12. If we let $E_a$ denote the saturated semi-continuum of $ax' - E$ containing $a$, we have $ab = \bar{E}_a + E + \bar{X}_b$, whence $ab - E \leq \bar{E}_a + \bar{X}_b$. If $E$ were a continuum of condensation of $ab$, we would have $ab = \overline{ab - E} \leq \bar{E}_a + \bar{X}_b$. Then

$$ab = \bar{E}_a + \bar{X}_b. \tag{1}$$

Now $\bar{E}_a < ax'$, since $x'$ is not a point of $\bar{E}_a$. But relation (1) shows that $\bar{E}_a \cdot \bar{X}_b \neq 0$. This is false, since, by the proof of § 15, $ax'$ is irreducible between $a$ and $\bar{X}_b$, and hence $\bar{E}_a$, being a real sub-continuum of $ax'$, cannot contain points of $\bar{X}_b$.

Therefore $E$ is not a continuum of condensation of $ab$.

17. THEOREM. *Let $x$ be a point of the limited irreducible continuum $ab$. Let the oscillatory set $X$ contain neither $a$ nor $b$. Let $x$ be a point of $\bar{X}_b$, but not of $\bar{X}_a$. Let $ax'$ be the sub-continuum of $ab$ irreducible between $a$ and $X \cdot \bar{X}_b$. Let the oscillatory set $E$ of $ax'$ about $x'$ be complete. Then the oscillatory set $X'$ of $ab$ about $x'$ is complete and contains $X$.*

PROOF. By § 7 $\bar{X}_b$ is irreducible between $x'$ and $b$. Let $F$ be the oscil-

latory set of $\bar{X}_b$ about $x'$. Then $F = X \cdot \bar{X}_b$ and, if $F_b$ denotes the saturated semi-continuum of $\bar{X}_b - F$ containing $b$, $F_b = X_b$ by § 9. Then

$$ab = ax' + \bar{X}_b = E_a + E + F + F_b,$$

where $E_a$ is the saturated semi-continuum of $ax' - E$ containing $a$. By § 8, Corollary,

$$X' = E + F.$$

As $ax'$ is irreducible between $a$ and $\bar{X}_b$, $E_a \cdot (F + F_b) = 0$. Also $E \cdot F_b = 0$, since $E < ax'$ and it was shown in the proof of § 15 that $X_b \cdot ax' = 0$.

As $E_a \cdot X' = E_a \cdot (E + F) = 0$, it is obvious that $X'_a \geq E_a$. If $X'_a > E_a$, we have $X'_a \cdot F_b \neq 0$. Now

$$E_a \cdot \bar{F}_b + \bar{E}_a \cdot F_b \leq E_a \cdot (F + F_b) + (E_a + E) \cdot F_b = 0.$$

Hence no sub-continuum of $ab$ joins $a$ and a point of $F_b$ and lies wholly in $E_a + F_b$. But $X'_a < E_a + F_b$. Thus the relation $X'_a \cdot F_b \neq 0$ is false, and $X'_a = E_a$. In like manner it can be shown that $X'_b = F_b$.

Then, since $x'$ is a point of both $\bar{X}'_a$ and $\bar{X}'_b$, $X'$ is complete. Since $x$ is a point of $\bar{X}_b$, it is a point of $X \cdot \bar{X}_b$. Since $F = X \cdot \bar{X}_b$, $x$ is a point of $X'$. Then by § 2 $(e)$, $X \leq X'$.

18. THEOREM. *Let $ab$ be a limited irreducible continuum. The necessary and sufficient condition for each point of $ab$ to lie on a complete oscillatory set is that $ab$ contains no indecomposible continuum which is not a continuum of condensation.*

PROOF. I. *It is sufficient.* We first observe that, if the oscillatory set $B$ about $b$ were not complete, then $B$ would be indecomposible and not a continuum of condensation by § 12. Likewise, the oscillatory set $A$ about $a$ is complete.

We now show that, if $A$ and $B$ are complete and $x$ lies on neither $A$ nor $B$, then neither $X_a$ nor $X_b$ is void. For, by § 2 $(e)$, $X < A_b$ and $X < B_a$. Hence $X \cdot A = X \cdot B = 0$ and $X$ contains neither $a$ nor $b$.

Let, then, $A$ and $B$ be complete and $x$ lie on neither set. If $x$ lies in both $\bar{X}_a$ and $\bar{X}_b$, $X$ is complete by definition. We cannot have neither $\bar{X}_a$ nor $\bar{X}_b$ containing $x$, for then, by § 13, $ab$ contains at least one indecomposible continuum which is not a continuum of condensation.

If $x$ is a point of $\bar{X}_b$, but not of $\bar{X}_a$, there is by § 15 an $x'$ on $X \cdot \bar{X}_b$ such that there is a unique $ax'$ and a unique $x'b = \bar{X}_b$. The oscillatory set

$E$ of $ax'$ about $x'$ must be complete, for if it were not the conditions of the hypothesis would be violated by § 16. Then by § 17 the oscillatory set $X'$ of $ab$ about $x'$ is complete and contains $X$. Precisely the same reasoning takes care of the case that $x$ is a point of $\bar{X}_a$, but not of $\bar{X}_b$.

Thus in every case $x$ lies on a complete oscillatory set.

II. *It is necessary.* Let $ab$ contain an indecomposible continuum $K$ which is not a continuum of condensation. Let $x$ be a point of $K$, but not a point of $\overline{ab - K}$. Then for $\delta$ sufficiently small, $V_\delta(x) < K$, and hence $X = K$.

If now $x$ were on some complete oscillatory set $X'$, we would have $K = X \leq X'$ by § 2 $(e)$. But then $\overline{ab - K} \geq \overline{ab - X'} = ab$, which contradicts the hypothesis regarding $K$.

COROLLARY. *For every oscillatory set of the limited irreducible continuum $ab$ to be complete or a part of a complete oscillatory set, it is necessary and sufficient that $ab$ contain no indecomposible continuum which is not a continuum of condensation.*

19. The previous theorem, together with § 6, shows that any continuum satisfying the hypotheses of § 18 is made up of complete oscillatory sets, no two of which have common points. Thus, if $X$ denotes any complete oscillatory set of $ab$ and $K = \{X\}$ is the aggregate of complete oscillatory sets of $ab$, we have

$$ab = \Sigma[X],$$

the summation sign being used in the restricted sense of § 3. For convenience we shall call such continua *separable*. Then we may restate § 18 as follows.

THEOREM. *The necessary and sufficient condition for the limited irreducible continuum $ab$ to be separable is that it contain no indecomposible continuum which is not a continuum of condensation.*

In a previous paper * it was shown that a *sufficient* condition for $ab$ to be separable is that the points of the first genre be everywhere dense in $ab$.

* See reference (E).

20. **A comparison of the oscillatory set with the notion of "prime part."** The definition of prime part, given by H. Hahn,* may be paraphrased as follows. Let $H$ be the aggregate of points of the second genre of the irreducible continuum $ab$. Then $\bar{H}$ is a closed set, composed of one or more components. Each component is a prime part. Also each point of $ab - \bar{H}$ is a prime part. These latter are of course points of the first genre which are not limiting points of $H$. If $\bar{H} = ab$, we call $ab$ a prime continuum. The author then proceeds to prove a set of theorems regarding prime parts, which are much like those of this paper.

From § 2 ($c$) and ($e$) it follows that every oscillatory set is a prime part or part of one; hence the oscillatory set is in general a smaller "element" than the prime part. That is effectively smaller is shown by the following example, given partly to illustrate this point and for later use.

Let $\{r_n\}$ denote the rational points in the interval $(-\pi, \pi)$ arranged in a sequence. Let $f_n(t) = 1/n^2 \sin \dfrac{1}{t - r_n}$, when $t \neq r_n$; $= 0$, when $t = r_n$. Let $F(t) = \sum_1^\infty f_n(t)$. Then $F(t)$ is continuous at each irrational point in the interval $(-\pi, \pi)$; and at a rational point $r_n$, $F(t)$ is discontinuous and has an oscillation of $2/n^2$ on the right and left of $t = r_n$.

Let $C$ be the aggregate composed of the graph of $F(t)$ and its limiting points. This is a limited continuum, irreducible between $a = (-\pi, F(-\pi))$ and $b = (\pi, F(\pi))$. The points of $C$ whose abscissas are rational are points of the second genre and are dense in $C$. Hence $C$ is a prime continuum. On the other hand the points whose abscissas are irrational are points of the first genre and these are dense in $C$; hence $C$ is separable. The complete oscillatory sets of $C$ are the points of the first genre and an enumerable set of linear segments of lengths $2/n^2$, $n = 1, 2, 3$, etc.

For an example where the points of the first genre are not everywhere dense see § 28.

*The correspondence between the complete oscillatory sets of a separable continuum and the points of a line.*

21. **The order relations of the aggregate of complete oscillatory sets.** Let $X$ be any complete oscillatory set of the separable (limited irreducible) continuum $ab$ and let $K = \{X\}$ be the aggregate whose elements are these complete oscillatory sets. As seen in § 19, $ab = \Sigma[X]$. The fact that $ab = X_a + X + X_b$, $X \cdot X_a = X \cdot X_b = X_a \cdot X_b = 0$ (See § 2 ($e$)) permits us

* See reference (C).

to order the aggregate $K$. For, if $Y$ is *another* element of $K$, we know that $Y < X_a$ if $y$ is a point of $X_a$ and that $Y < X_b$ if $y$ is a point of $X_b$. In the former case we say that $Y$ precedes $X$; in the latter that $Y$ follows $X$.

It is a simple matter to show that these relations are reversible and mutually exclusive, and that, if $X$, $Y$, and $Z$ are three different elements of $K$ such that $X$ precedes $Y$ and $Y$ precedes $Z$, then $X$ precedes $Z$. Thus $K$ is a simply ordered aggregate. Furthermore $K$ has a first and a last element, namely the complete oscillatory sets $A$ and $B$ respectively.

The next few sections are devoted to showing that the order type of $K$ is that of a closed linear continuum. In one of the papers referred to above * this was done for the case that the points of the first genre are everywhere dense in $ab$. Further investigation has shown that this restriction is not necessary. The line of attack is the same as in the special case previously treated and in two theorems the proofs used need no alteration. These will therefore be omitted.

22. THEOREM. *Let the limited irreducible continuum $ab$ be separable. Then the aggregate of complete oscillatory sets of $ab$ has a dense order type.*

For the proof of this theorem see reference (E), § 19.

23. THEOREM. *Let the limited irreducible continuum $ab$ be separable and let $K$ be the aggregate of complete oscillatory sets of $ab$. Then there is an enumerable sub-set $K'$ of $K$ which contains neither $A$ nor $B$ such that if $E_k$ is any element of $K'$ and $U = U[E_k]$, then the point-set $U$ is dense in $ab$.*

PROOF. Let $K = \{X\}$. Let $\epsilon_1 > \epsilon_2 > \epsilon_3 > \cdots$ and $\epsilon_i \to 0$. Then the points in space whose distances from a particular $X$ are less than $\epsilon_1$ constitute a region $R$. Obviously $ab < U[R]$. By the Borel theorem there is a finite set $R_1, R_2, \cdots, R_{n_1}$ of these regions such that $ab < R_1 + R_2 + \cdots + R_{n_1}$. In the same way there is a finite set of regions $R_{n_1+1}, R_{n_1+2}, \cdots, R_{n_2}$ such that $ab < R_{n_1+1} + R_{n_1+2} + \cdots + R_{n_2}$ and for each $R_{n_1+i}$, $(1 \leqq i \leqq n_2 - n_1)$ there is an $X_{n_1+i}$ such that all the points of $R_{n_1+i}$ have a distance from $X_{n_1+i}$ less than $\epsilon_2$. This process carried out for each $\epsilon_i$ yields an enumerable set of regions which may be denoted by $\{R_k\}$. Associated with each $R_k$ is a complete oscillatory set $X$ contained within it, which may be denoted

* See reference (E).

by $E_k$ and the distance of each point of $R_k$ from $E_k$ is less than some $\epsilon_i$. Set $K_1 = \{E_k\}$, and $U_1 = U[E_k]$.

Now let $x$ be any point of $ab$ not in $U_1$. Take any $\delta > 0$ and let $\epsilon_i < \delta$. The definition of the sets $\{R_k\}$ in the previous paragraph shows that $x$ lies in some $R_k$ such that the distance of every point of $R_k$ from the corresponding set $E_k$ is less than $\epsilon_i$. Hence the distance of $x$ from the set $E_k$ is less than $\epsilon_i$. But then $E_k \cdot V_\delta(x) \neq 0$. Therefore $U_1 \cdot V_\delta(x) \neq 0$, which shows that $U_1$ is dense in $ab$.

If neither $A$ nor $B$ is an element of $K_1$, $K_1$ and $U_1$ are the $K'$ and $U$ required in the theorem. In the contrary case let $K' = K_1 - (A + B)$ and $U = U_1 - (A + B)$. Obviously $K'$ is enumerable. For any $x$ in $ab$ and any $\delta > 0$, $V_\delta(x)$ contains a point $z$ not in either $A$ or $B$, since $A$ and $B$ are continua of condensation. Let $\eta > 0$ be less than $\delta$ and less than $Dist(z,\ A + B)$. Since $U_1$ is dense in $ab$, $U_1 \cdot V_\eta(z) \neq 0$; whence $U \cdot V_\eta(z) \neq 0$. Hence, if $\sigma = 2\delta$, $U \cdot V_\sigma(x) \neq 0$ and $U$ is dense in $ab$.

24. THEOREM. *Let the limited irreducible continuum $ab$ be separable and let $K$ be the aggregate of complete oscillatory sets of $ab$. Then there is an open enumerable sub-set $K'$ of $K$ which is dense in $K$ and dense in itself.*

PROOF. Let $K'$ be defined as in § 23. Since $K'$ contains neither $A$ nor $B$, it is open. Let $X$ and $Y$ be elements of $K$ and $X$ precede $Y$. By § 22 there is an element $Z$ such that $Z$ follows $X$ and precedes $Y$. Then $Z \cdot (X_a + X) = Z \cdot (Y + Y_b) = 0$. Let $z$ be a point of $Z$. For a positive $\delta$ small enough $V_\delta(z) < X_b$ and $V_\delta(z) < Y_a$.

By § 23 $V_\delta(z)$ contains a point $e$ of some element $E_k$ of $K'$. Since $z$ is a point of $X_b$, $E_k < X_b$; likewise $E_k < Y_a$. Therefore $E_k$ follows $X$ and precedes $Y$.

Thus $K'$ is dense in $K$ and, since $K$ is dense in itself, so is $K'$.

25. THEOREM. *Let the limited irreducible continuum $ab$ be separable and let $K$ be the aggregate of complete oscillatory sets of $ab$. Then the order type of $K$ is that of a finite closed segment.*

The proof is verbatim that of § 21 in reference (E).

26. **Supra-continuous functions.*** If we let $T$ denote the closed in-

* The functions defined in this section have a number of interesting properties which have been worked out by L. S. Hill in some as yet unpublished work. See also R. L. Moore, "Concerning upper semi-continuous collections of continua, etc.," *Proc. Nat. Acad. of Sciences*, Vol. 10, No. 8 (1924).

terval $0 \leqq t \leqq 1$, the preceding theorem gives a uniform correspondence between the points of $T$ and the complete oscillatory sets $X$ of $K$. This we denote by $X = f(t)$.

In this relation an aggregate is a function of a point, a generalization of the ordinary case where a point is a function of a point. The following easy generalizations of the notions of the upper and lower closed limits * of a sequence of aggregates permits the definition of continuity for such a function.

Let $\tau$ be a point of $T$. Let $x$ be a point such that for every $\epsilon > 0$ and every $\delta > 0$, $V_\epsilon(x)$ contains a point of $f(t)$ for some $t$ in $V_\delta(\tau)$. If $f(\tau)$ is the class of points $x$ satisfying this relation, we say that $f(\tau)$ is the *upper limit* of $f(t)$ as $t \to \tau$ and write $f(\tau) = \overline{\lim_{t \to \tau}} f(t)$.

Let $x$ be a point such that for every $\epsilon > 0$ there is some $\delta > 0$ for which $V_\epsilon(x)$ contains a point of $f(t)$ for *every* $t$ in $V_\delta(\tau)$. If $f(\tau)$ is the class of points satisfying this relation, we say that $f(\tau)$ is the *lower limit* of $f(t)$ as $t \to \tau$ and write $f(\tau) = \underline{\lim_{t \to \tau}} f(t)$.

If $f(\tau) = \overline{\lim}\ f(t)$, $f(t)$ is said to be *supra-continuous* at $t = \tau$. If $f(\tau) = \underline{\lim}\ f(t)$, $f(t)$ is said to be *infra-continuous* at $t = \tau$. If $f(t)$ is both supra- and infra-continuous at $t = \tau$, it is said to be *continuous* at this point. When $f(\tau)$ consists of a single point, the first and last definitions coincide with the ordinary concept of continuity at a point.

27. THEOREM. *Let the limited irreducible continuum $ab$ be separable and let $K = \{X\}$ be the aggregate of complete oscillatory sets of $ab$. Let $T = \{t\}$, where $0 \leqq t \leqq 1$.† Then there is a uniform correspondence $X = f(t)$ between the elements of $K$ and the points of $T$ and $f(t)$ is supra-continuous at each point of $T$.*

PROOF. The first part of the theorem is a mere re-statement of § 25.

Let $\tau$ be an inner point of $T$, let $C = f(\tau)$, and let $C$ be the oscillatory set of $ab$ about $c$. Since $C$ is complete, it follows from § 5 that for every $\epsilon > 0$, there is an irreducible sub-continuum $xy$ of $ab$, where $x$ is a point of $C_a$ and $y$ is a point of $C_b$, such that

$$C < xy < V_\epsilon(C). \tag{1}$$

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* (G) F. Hausdorff, *Grundzüge der Mengenlehre*, p. 236.

† In the following proof the symbol $<$, when used in connection with values of $t$, has of course its ordinary meaning.

Let $X'$ and $Y'$ be the complete oscillatory sets containing $x$ and $y$, respectively. Since $X'$, $Y'$, and $C$ are complete, we see from § 2 $(e)$ that

$$ab = (X'_a + X') + X'_b \cdot C_a + C + Y'_a \cdot C_b + (Y' + Y'_b),$$

where no two of the sets on the right have common points and none is void. Then

$$X'_b \cdot C_a + C + Y'_a \cdot C_b < xy. \tag{2}$$

Now let $X' = f(t')$ and $Y' = f(t'')$. Since $K$ and $T$ are similarly ordered, $t' < \tau < t''$. Let $t$ be any point such that $t' < t < t''$ and let $X = f(t)$. If $t = \tau$, $X = C$. If $t' < t < \tau$, $X < X'_b$ and $X < C_a$; hence $X < X'_b \cdot C_a$. If $\tau < t < t''$, $X < Y'_a \cdot C_b$. Thus in any case

$$X < X'_b \cdot C_a + C + Y'_a \cdot C_b.$$

This with relations (1) and (2) gives $X < V_\epsilon(C)$. Hence

$$\overline{\lim_{t \to \tau}} f(t) \leq C. \tag{3}$$

If $\xi$ is any point of $C$, there is a sequence $\{x_n\}$ of points converging to $\xi$ but not lying in $C$, as $C$ is a continuum of condensation. There is no loss in generality if we assume that these all lie in one of the sets $C_a$ or $C_b$, say in $C_a$. Each $x_n$ lies on some complete oscillatory set $X_n = f(t_n)$. Since $X_n < C_a$, $t_n < \tau$. Let $\tau'$ be a point of accumulation of the points $t_n$; then $\tau' \leqq \tau$. Since $\tau'$ is a point of accumulation of $\{t_n\}$ there is a partial sequence of points $\{t_\nu\}$, either identical with $\tau'$ or converging to it. Suppose that $\tau' < \tau$ and $C' = f(\tau')$. By relation (3)

$$C' \geq \overline{\lim_{t_\nu \to \tau'}} f(t_\nu) = \overline{\lim} X_\nu.$$

Then $\xi$ is a point of $C'$, which is contrary to the hypothesis regarding $\xi$, since $C' \cdot C = 0$. Hence $\tau' = \tau$ and $t_\nu \to \tau$. In this case no $t_\nu = \tau$. Since $\xi = \lim x_\nu$, it is a point of $\overline{\lim} X_\nu = \overline{\lim_{t_\nu \to \tau}} f(t_\nu)$. Therefore

$$C \geq \overline{\lim_{t \to \tau}} f(t). \tag{4}$$

Relations (3) and (4) give the theorem for this case. If $\tau = 0$ or $\tau = 1$, the same result is obtained by an obvious modification of the above proof.

28. **An example.** When every point of the limited irreducible continuum $ab$ is of the first genre we have ordinary continuity, for each $X$ is a single point, and the correspondence of § 27 becomes the homeomorphic correspondence whose existence was proved by Janiszewski.

It can be shown * that supra-continuous functions of the type here used have an everywhere dense set of points of continuity. The continuum defined in § 20 is an example of this kind. At each rational point of $T = (-\pi \leqq t \leqq \pi)$, $X = f(t)$ is discontinuous; at each irrational point it is continuous. In this case at each point of continuity $X$ reduces to a single point, which is of the first genre. From this example it might be inferred that, if $ab$ is separable, the function $X = f(t)$ is continuous only for values of $t$ for which $X$ is a single point and hence that a separable continuum must have an everywhere dense set of points of the first genre. That this is not true is shown by the following example.

This is based on the Cantor set. Let $ab$ and $cd$ be opposite sides of a parallelogram. The continuum consists of the sides $ac$ and $bd$, of an enumerable set of broken lines $\{L_n\}$ described below, and of the limiting points of these lines, which constitute a non-enumerable set of straight lines joining $ab$ and $cd$. The broken lines are defined as follows:

$L_1$ is the pair of lines joining the mid-point $f_1$ of $cd$ to the points $e_1$ and $e_1'$ trisecting $ab$.

$L_2$ is the pair of lines joining the mid-point $e_2$ of $ae_1$ to the points $f_2$ and $f_2'$ trisecting $cf_1$.

$L_3$ is the pair of lines joining the mid-point $e_3$ of $e_1'b$ to the points $f_3$ and $f_3'$ trisecting $f_1d$.

Repeating this construction in the four parallelograms $ae_2f_2c$, $e_2e_1f_1f'_2$, $e'_1e_3f_3f_1$, $e_3bdf'_3$ gives twelve more broken lines. Thus $L_4$ is the pair of lines joining the mid-point $f_4$ of $cf_2$ to the points $e_4$ and $e'_4$ trisecting $ae_2$; $L_5$ is the pair of lines joining the mid-point $e_5$ of $ae_4$ to the points $f_5$ and $f'_5$ trisecting $cf_4$; etc. Let this be continued indefinitely.

It is easily seen that the set as defined above is a continuum irreducible between every point on $aa'$ and every point on $bb'$. The complete oscillatory sets are the lines $aa'$ and $bb'$, the broken lines, and the limiting lines. Thus *every point is of the second genre,* and has an oscillation greater than or equal to the perpendicular distance between the lines $ab$ and $a'b'$. The continuum is obviously separable.

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* See footnote under § 26.

If we let $X = f(t)$ as in § 27, $f(t)$ is continuous as points for which the corresponding $X$ is a limiting line; it is discontinuous at points where the corresponding $X$ is a broken line.

This example suggests the further question as to whether we can have a separable continuum for which $X = f(t)$ is continuous at every point of a subinterval of $T$ but reduces to a point nowhere in this interval. If such a continuum exists, it should have interesting properties.

NEW HAVEN, CONN.

# The Expansion Problem for a Certain System of Ordinary Linear Second Order Differential Equations.

By M. G. Carman.

## 1. The Formal Expansions.

With the system of $n$ equations

$$(1) \qquad L(y_s) \equiv d^2y_s/dx^2 + \rho^2 y_s = 0, \qquad s = 1, 2, \cdots, n,$$

we associate $2n$ linear homogeneous conditions in $y_1(a)$, $y_2(a)$, $\cdots$, $y_n(a)$, $y_1'(a)$, $y_2'(a)$, $\cdots$, $y_n'(a)$, $y_1(b)$, $y_2(b)$, $\cdots$, $y_n(b)$, $y_1'(b)$, $y_2'(b)$, $\cdots$, $y'_n(b)$,

$$(2) \qquad Y_1(y) = 0, \quad Y_2(y) = 0, \cdots, \quad Y_{2n}(y) = 0,$$

and with the adjoint system

$$(3) \qquad M(z_s) \equiv d^2z_s/dx^2 + \rho^2 z_s = 0, \qquad s = 1, 2, \cdots, n,$$

we associate $2n$ like adjoint conditions

$$(4) \qquad Z_{2n+1}(z) = 0, \quad Z_{2n+2}(z) = 0, \cdots, \quad Z_{4n}(z) = 0.$$

If systems (1) and (3) are written for the same value of $\rho^2$, we have

$$(5) \qquad \int_a^b \sum_{j=1}^{n} [z_j L(y_j) - y_j M(z_j)]\,dx = \sum_{j=1}^{n} (z_j y_j' - y_j z_j') \Big|_{x=a}^{x=b}$$

$$= \sum_{j=1}^{4n} Y_j(y) Z_j(z) = 0,$$

where the $Y_j$ and $Z_j$ are linear and homogeneous functions in variables $y$ and $z$ respectively. If the forms $Y_j$ are chosen so that they are linearly independent, then the $Z_j$ are uniquely determined and are linearly independent. Conditions (2) and (4) are obtained by choosing the $Y_j$ and $Z_j$ in this manner. The relation

$$(6) \qquad \sum_{j=1}^{n} (z_j y_j' - y_j z_j') \Big|_{x=a}^{x=b} = 0$$

holds, then, whether the sets $y_1, y_2, \cdots, y_n$ and $z_1, z_2, \cdots, z_n$ are solutions of (1) and (3) respectively for the same or for different values of $\rho^2$, so long as the first set satisfies conditions (2) and the second set conditions (4).

THEOREM 1. *If, for $\rho = \bar{\rho}$, a set of solutions* $\bar{y}_1(x), \bar{y}_2(x), \cdots, \bar{y}_n(x)$ of (1), (2) exists, then a set of solutions $\bar{z}_1(x), \bar{z}_2(x), \cdots, \bar{z}_n(x)$ of (3), (4) exists for $\rho = \bar{\rho}$; if the set $\bar{y}_1(x), \bar{y}_2(x), \cdots, \bar{y}_n(x)$ is unique except for a constant factor, then the set $\bar{z}_1(x), \bar{z}_2(x), \cdots, \bar{z}_n(x)$ is unique except for a constant factor.*

A proof of this theorem is, with slight modifications, the same as a proof given by Birkhoff † of a similar theorem for the case of a single equation and associated boundary conditions.

A *characteristic value* of $\rho$ is a value for which the system (1), (2) has a solution. If this solution is unique, except for a constant factor, then the characteristic value is *simple.*

Let $\rho = \rho_\lambda$ and $\rho = \rho_\mu$ be two characteristic values. Then we have

$$\int_a^b \sum_{j=1}^n [z_j L(y_j) - y_j M(z_j)]\, dx = \sum_{j=1}^n (z_j y_j' - y_j z_j') \Big|_{x=a}^{x=b}$$

$$+ (\rho_\lambda^2 - \rho_\mu^2) \int_a^b \sum_{j=1}^n y_j z_j dx = 0,$$

where $y_j$ is formed for $\rho = \rho_\lambda$ and $z_j$ for $\rho = \rho_\mu$. Recalling (6), one sees that the following theorem is proved.

THEOREM 2. *If $\rho_\lambda$ and $\rho_\mu$ are characteristic values such that $\rho_\lambda^2 \neq \rho_\mu^2$ and if $y_1(\rho_\lambda, x), y_2(\rho_\lambda, x), \cdots, y_n(\rho_\lambda, x)$ is a set of solutions of (1), (2) for $\rho = \rho_\lambda$ and $z_1(\rho_\mu, x), z_2(\rho_\mu, x), \cdots, z_n(\rho_\mu, x)$ is a set of solutions of (3), (4) for $\rho = \rho_\mu$,* then

$$\int_a^b \sum_{j=1}^n y_j(\rho_\lambda, \xi) z_j(\rho_\mu, \xi)\, d\xi = 0.$$

We are now in a position to determine formally the coefficients $c_k$ of the formal expansions

$$f_s(x) = \sum_{k=1}^\infty c_k y_s(\rho_k, x), \qquad s = 1, 2, \cdots, n, \tag{7}$$

where, if $\rho_k$ is not zero, it satisfies the relation

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* The set $y_1 \equiv 0, y_2 \equiv 0, \ldots, y_n \equiv 0$ is always excluded.

† *Transactions of the American Mathematical Society*, 9 (1908), 375-376.

$$-\pi/2 < \arg \rho_k \leqq \pi/2.$$

Multiplying both members of (7) by $z_s(\rho_k, x)$, summing as to $s$ from 1 to $n$, and integrating with respect to $x$ from $a$ to $b$, we have formally

$$c_k = \frac{\int_a^b \sum_{j=1}^{n} z_j(\rho_k, \xi) f_j(\xi)\, d\xi}{\int_a^b \sum_{j=1}^{n} z_j(\rho_k, \xi) y_j(\rho_k, \xi)\, d\xi} \qquad (8)$$

The principal object of this paper is to establish the validity of the expansions (7), (8) with suitable restrictions on the functions $f_s(x)$ and conditions (2). The method employed consists in the use of a contour integral to represent the sum of $m$ terms of one of the expansions and in the determination of the limit of this integral as $m$ becomes infinite. This powerful method, due to Birkhoff,* was first used in the case of the expansion of a single function. It has since been extended to the problem of the simultaneous expansion of $n$ functions, but only in connection with systems of first order equations.†

## 2. The Characteristic Values.

If we write conditions (2) explicitly as

$$Y_r(y) \equiv \sum_{j=1}^{n} [\alpha_{rj} y_j(a) + \beta_{rj} y'_j(a) + \gamma_{rj} y_j(b) + \delta_{rj} y'_j(b)] = 0, \qquad (9)$$
$$r = 1, 2, \cdots, 2n,$$

we may readily prove the following theorem.

Theorem 3. *A necessary and sufficient condition that $\rho = \bar{\rho}$ is a characteristic value, different from zero, is that $\Delta(\rho) = 0$, where $\Delta(\rho)$ is the $2n$-rowed determinant of which the element in the $r$th row and $m$th column is*

$$\alpha_{rm} \cos \rho a + \gamma_{rm} \cos \rho b - \rho(\beta_{rm} \sin \rho a + \delta_{rm} \sin \rho b), \text{ if } 1 \leqq m \leqq n,$$
$$\alpha_{rm} \sin \rho a + \gamma_{rm} \sin \rho b + \rho(\beta_{rm} \cos \rho a + \delta_{rm} \cos \rho b), \text{ if } n+1 \leqq m \leqq 2n.$$

*A characteristic value of $\rho = \bar{\rho}$ is simple when and only when some first minor of $\Delta(\bar{\rho})$ does not vanish.*

For if we substitute $a_j \cos \rho x + b_j \sin \rho x$ for $y_j$ in (9), we obtain a sys-

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* *Loc. cit.*

† Among the most important papers on this subject are those of Birkhoff and Langer, *Proceedings of the American Academy of Arts and Sciences*, 58 (1923), 51-128, and Schur, *Mathematische Annalen*, 82 (1920-21), 213-236, where further references are given.

tem of $2n$ linear homogeneous equations in the $2n$ quantities $a_j$, $b_j$, and the determinant of this system is $\Delta(\rho)$.

The approximate situation of the roots of $\Delta(\rho) = 0$ is desired for large values of $|\rho|$.

By breaking $\Delta(\rho)$ up into a sum of determinants which have as elements single terms instead of the sum of four terms and from the columns of which a trigonometric function or a trigonometric function multiplied by $\rho$ may be factored, one sees that the equation $\Delta(\rho) = 0$ takes the form

$$(10) \qquad \rho^{2n} |\beta, \delta| \sin^n \rho(b-a) + K_{2n-1}\rho^{2n-1} + \cdots + K_1\rho + K_0 = 0,$$

where

$$|\beta, \delta| = \begin{vmatrix} \beta_{11} & \cdots & \beta_{1n} & \delta_{11} & \cdots & \delta_{1n} \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \cdot & & \cdot & \cdot & & \cdot \\ \beta_{2n,1} & \cdots & \beta_{2n,n} & \delta_{2n,1} & \cdots & \delta_{2n,n} \end{vmatrix},$$

and where the $K_s$ are polynomials in $\cos \rho(b-a)$, $\sin \rho(b-a)$ of degree not greater than $n$. We shall suppose that $|\beta, \delta| \neq 0$. By dividing both members of (10) by $\rho^{2n-1}\sin^n \rho(b-a)$ one sees that, since

$$|\cos \rho(b-a)/\sin \rho(b-a)|$$

is bounded as the imaginary part of $\rho$ becomes infinite, a constant $R_1$ exists such that, if $\bar{\rho}$ is a root of $\Delta(\rho) = 0$, the imaginary part of $\bar{\rho}$ is less than $R_1$ in absolute value. All the roots of $\Delta(\rho) = 0$ therefore lie in a region $S$ of the $\rho$-plane which consists of a strip of width $2R_1$ whose center line is the axis of reals. If we divide both members of (10) by $\rho^{2n}|\beta, \delta|$, we have

$$\sin^n \rho(b-a) = g(\rho),$$

where $g(\rho)$ tends to zero in $S$ with $1/\rho$. Hence, given a small positive constant $\epsilon$, a constant $R_2$ may be found such that, for $|\rho| > R_2$, the only roots of $\Delta(\rho) = 0$ lie in circles of radius $\epsilon$ with centers at the points $k\pi/(b-a)$, where $k$ assumes all integral values for which $|k\pi/(b-a)| > R_2$. Let $\Gamma$ be one of these circles. Then the number of roots of $\Delta(\rho) = 0$ within $\Gamma$ is given by

$$I = 1/2\pi i \int_\Gamma \Delta'(\rho)/\Delta(\rho)\, d\rho = 1/2\pi i \int_\Gamma \left[\frac{n(b-a)\cos \rho(b-a)}{\sin \rho(b-a)} + B(\rho)/\Delta(\rho)\right] d\rho,$$

where, if $\Gamma$ is taken sufficiently far out on the axis of reals,

$$|B(\rho)/\Delta(\rho)| < 1/\epsilon.$$

Hence we have

$$n - \delta \leqq |\, I \,| \leqq n + \delta, \qquad \delta < 1,$$

But since I must be equal to an integer, $I = n$. This result may be stated as follows.

THEOREM 4. *If the determinant* $|\, \beta, \delta \,| \neq 0$, *then corresponding to an arbitrarily small positive constant* $\epsilon$ *there exists a constant* $R$, *such that, for* $|\, \rho \,| > R$, *all the roots of* $\Delta(\rho) = 0$ *lie within circles of radius* $\epsilon$ *with centers at the points* $k\pi/(b-a)$, *where* $k$ *assumes all integral values for which* $|\, k\pi/(b-a) \,| > R$, *and such that in each of these circles the equation* $\Delta(\rho) = 0$ *has exactly* $n$ *roots.*

## 3. THE GREEN'S FUNCTIONS.

The $n^2$ Green's functions $G_{st}(x, \xi; \rho^2)$, $s, t = 1, 2, \cdots, n$, are defined as follows: the set of functions $G_{st}(x, \xi; \rho^2)$, $s = 1, 2, \cdots, n$, shall satisfy conditions (9) for each value of $t$ from 1 to $n$; each such set shall satisfy equations (1) except at one point $x = \xi$, $a < \xi < b$, where each of the functions $G_{st}(x, \xi; \rho^2)$ shall be continuous and where all except the $n$ functions $G_{tt}(x, \xi; \rho^2)$ shall have continuous first derivatives with respect to $x$; at $x = \xi$ the first derivative of each of the $n$ functions $G_{tt}(x, \xi; \rho^2)$ shall have a jump of one—that is,

$$G'_{tt}(\xi + 0, \xi; \rho^2) - G'_{tt}(\xi - 0, \xi; \rho^2) = 1, \qquad t = 1, 2, \cdots, n.$$

If we let

$$G_{st}(x, \xi; \rho^2) = \begin{cases} A_{st} \cos \rho x + B_{st} \sin \rho x, & a \leqq x < \xi, \\ A'_{st} \cos \rho x + B'_{st} \sin \rho x, & \xi < x \leqq b, \end{cases}$$

where $t$ is constant and $s$ runs over the set $1, 2, \cdots, n$, the conditions named are sufficient to determine the $4n$ coefficients $A_{st}, B_{st}, A'_{st}, B'_{st}$ uniquely. In fact we find that

$$\rho G_{st}(x, \xi; \rho^2) = \frac{\Delta_{st}(\rho, \xi) \cos \rho x + \Delta_{n+s,t}(\rho, \xi) \sin \rho x}{\Delta(\rho)}$$

$$= \frac{D_{st}(\rho, \xi) \cos \rho x + D_{n+s,t}(\rho, \xi) \sin \rho x}{\Delta(\rho)}, \quad s \neq t,$$

$$\rho G_{tt}(x, \xi; \rho^2) = \begin{cases} \dfrac{\Delta_{tt}(\rho, \xi) \cos \rho x + \Delta_{n+t,t}(\rho, \xi) \sin \rho x}{\Delta(\rho)}, & a \leqq x < \xi, \\ \dfrac{D_{tt}(\rho, \xi) \cos \rho x + D_{n+t,t}(\rho, \xi) \sin \rho x}{\Delta(\rho)}, & \xi < x \leqq b, \end{cases}$$

where $\Delta_{mt}(\rho, \xi)$ is obtained from $\Delta(\rho)$ by replacing the $r$th element, $r = 1, 2, \cdots, 2n$, in the $m$th column by

$$\gamma_{rt} \sin \rho(\xi - b) - \rho\delta_{rt} \cos \rho(\xi - b),$$

and where $D_{mt}(\rho, \xi)$ is obtained from $\Delta(\rho)$ by replacing the $r$th element, $r = 1, 2, \cdots, 2n$, in the $m$th column by

$$-\alpha_{rt} \sin \rho(\xi - a) + \rho\beta_{rt} \cos \rho(\xi - a).$$

We shall now establish two theorems concerning the Green's functions which will be needed later.

THEOREM 5. *If $r_1(x), r_2(x), \cdots, r_n(x)$ are continuous functions of $x$ in the interval $(ab)$, then the system*

$$(11) \qquad \begin{aligned} &d^2y_s/dx^2 + \rho^2 y_s = r_s(x), \qquad s = 1, 2, \cdots, n, \\ &Y_1(y) = 0, \quad Y_2(y) = 0, \cdots, \quad Y_{2n}(y) = 0, \end{aligned}$$

*has the unique solution*

$$(12) \qquad y_s(x) = \int_a^b \sum_{t=1}^{n} r_t(\xi) G_{st}(x, \xi; \rho^2)\, d\xi, \qquad s = 1, 2, \cdots, n,$$

*provided the value of $\Delta(\rho)$ is different from zero.*

If we put

$$y_s = \int_a^b r_s(\xi) G_{ss}(x, \xi; \rho^2)\, d\xi,$$

then

$$dy_s/dx = \int_a^b r_s(\xi)\ d/dx\ G_{ss}(x, \xi;\ \rho^2)\, d\xi,$$

$$\begin{aligned} d^2y_s/dx^2 &= \int_a^b r_s(\xi)\ d^2/dx^2\ G_{ss}(x, \xi; \rho^2)\, d\xi \\ &\qquad + [r_s(\xi)\ d/dx\ G_{ss}(x, \xi; \rho^2)]_{\xi=x+0}^{\xi=x-0} \\ &= \int_a^b r_s(\xi)\ d^2/dx^2\ G_{ss}(x, \xi; \rho^2)\, d\xi + r_s(x). \end{aligned}$$

Substituting these values for $y_s$ and $d^2y_s/dx^2$ in (11) we see that the differential equations are satisfied. Likewise we see that the $s$th equation of the corresponding system of homogeneous equations is satisfied by

$$y_s(x) = \int_a^b r_t(\xi) G_{st}(x, \xi; \rho^2)\, d\xi, \qquad t \neq s,$$

since $G_{st}(x, \xi; \rho^2)$ has a continuous first derivative if $s$ and $t$ are different. The differential equations therefore have for a solution the set (12). This set, moreover, satisfies the conditions $Y_j(y) = 0$ and hence is a solution of system (11). This solution is unique; for, if $\Delta(\rho) \neq 0$, the conditions

$Y_j(y) = 0$ are sufficient to determine uniquely the arbitrary constants which appear in the general solution of the differential equations.

THEOREM 6. *If the functions* $H_{st}(x, \xi; \rho^2)$, $s$, $t = 1, 2, \cdots, n$, *are the Green's functions for the adjoint problem* (3), (4), *then*

$$G_{st}(x, \xi; \rho^2) = H_{ts}(\xi, x; \rho^2).$$

For any $\alpha$ and $\beta$ we have

$$\int_\alpha^\beta \sum_{j=1}^n [z_j L(y_j) - y_j M(z_j)]\,dx = \sum_{j=1}^n [z_j y'_j - y_j z'_j]_{x=\alpha}^{x=\beta} = 0, \tag{13}$$

if the $y_j$ and $z_j$ are solutions of systems (1) and (3) respectively in the interval $(\alpha\beta)$ for the same value of $\rho^2$. Choose $\xi_1$ and $\xi_2$ so that $a < \xi_1 < \xi_2 < b$. In (13) put $y_j = G_{jt}(x, \xi_1; \rho^2)$ and $z_j = H_{js}(x, \xi_2; \rho^2)$ and let $\alpha$, $\beta$ have successively the values $a$, $\xi_1 - \epsilon$; $\xi_1 + \epsilon$, $\xi_2 - \epsilon$; $\xi_2 + \epsilon$, $b$. Adding the three relations thus obtained and letting $\epsilon$ go to zero, we have, considering the properties by means of which the Green's functions were defined,

$$-H_{ts}(\xi_1, \xi_2, ; \rho^2) + G_{st}(\xi_2, \xi_1; \rho^2) = 0.$$

We should obtain the same result for $\xi_1 \geqq \xi_2$ and hence the theorem is true.*

## 4. Representation of the Formal Expansions as Contour Integrals.

If $R_{st}(x, \xi; \rho_k)$ denotes the residue of $\rho G_{st}(x, \xi; \rho^2)$ at $\rho = \rho_k$, where $\rho_k$ is a simple characteristic value for which $\rho G_{st}(x, \xi; \rho^2)$ has a simple pole, then the set of functions $R_{st}(x, \xi; \rho_k)$, for $t$ fixed and $s$ running over the set $1, 2, \cdots, n$, obviously satisfies equations (1). It is easy to verify that the same set also satisfies conditions (9). Then if $y_1(\rho_k, x)$, $y_2(\rho_k, x)$, $\cdots$, $y_n(\rho_k, x)$ is a set of solutions of (1), (9) for $\rho = \rho_k$, we have

$$R_{st}(x, \xi; \rho_k) = A_t y_s(\rho_k, x), \tag{14}$$

where, from the definition of simple characteristic value, $A_t$ is independent of $s$ and $x$. From Theorem 6 it follows that

$$R_{st}(x, \xi; \rho_k) = B_s z_t(\rho_k, \xi), \tag{15}$$

where $B_s$ is independent of $\xi$ and $t$ and where $z_1(\rho_k, \xi)$, $z_2(\rho_k, \xi)$, $\cdots$, $z_n(\rho_k, \xi)$ is a set of solutions of (3), (4) in $\xi$ for $\rho = \rho_k$. Combining (14) and (15) we have

$$R_{st}(x, \xi; \rho_k) = C y_s(\rho_k, x) z_t(\rho_k, \xi),$$

* Cf. Bôcher, *Leçons sur les Méthodes de Sturm*, p. 104.

where $C$ is independent of $x, \xi, s, t$.

Since $\rho = \rho_k$ is assumed to be a simple pole of $\rho G_{st}(x, \xi; \rho^2)$, we have

$$\lim_{\rho=\rho_k} [(\rho - \rho_k)\rho G_{st}(x, \xi; \rho^2) - C y_s(\rho_k, x) z_t(\rho_k, \xi)] = 0.$$

Multiplying both members of this equation by $y_t(\rho_k, \xi)$, summing as to $t$ from 1 to $n$, and integrating with respect to $\xi$ from $a$ to $b$, one finds

$$\lim_{\rho=\rho_k} \left[\frac{\rho}{\rho + \rho_k} \int_a^b \sum_{t=1}^n (\rho^2 - \rho^2_k) G_{st}(x, \xi; \rho^2) y_t(\rho_k, \xi)\, d\xi - C y_s(\rho_k, x) \int_a^b \sum_{t=1}^n y_t(\rho_k, \xi) z_t(\rho_k, \xi)\, d\xi\right] = 0. \tag{16}$$

Now since

$$(\rho^2 - \rho^2_s) y_k(\rho_k, x) = \frac{d^2 y_s(\rho_k, x)}{dx^2} + \rho^2 y_s(\rho_k, x),$$

it follows from Theorem 5 that

$$y_s(\rho_k, x) = \int_a^b \sum_{t=1}^n (\rho^2 - \rho_k^2) G_{st}(x, \xi; \rho^2) y_t(\rho_k, \xi)\, d\xi.$$

Equation (16) then becomes

$$\lim_{\rho=\rho_k} \left[\frac{\rho}{\rho + \rho_k} y_s(\rho_k, x) - C y_s(\rho_k, x) \int_a^b \sum_{t=1}^n y_t(\rho_k, \xi) z_t(\rho_k, \xi)\, d\xi\right] = 0,$$

whence it follows that

$$C = \frac{K}{\displaystyle\int_a^b \sum_{t=1}^n y_t(\rho_k, \xi) z_t(\rho_k, \xi)\, d\xi},$$

where

$$K = \begin{cases} 1, & \text{if } \rho_k = 0, \\ \frac{1}{2}, & \text{if } \rho_k \neq 0. \end{cases}$$

We have, then,

$$\frac{K y_s(\rho_k, x) z_j(\rho_k, \xi)}{\displaystyle\int_a^b \sum_{t=1}^n y_t(\rho_k, \xi) z_t(\rho_k, \xi)\, d\xi} = 1/2\pi i \int_{\gamma_k} \rho G_{sj}(x, \xi; \rho^2)\, d\rho, \tag{17}$$

where $\gamma_k$ is a contour in the $\rho$-plane inclosing just the characteristic value $\rho_k$.

Multiplying both members of (17) by $f_j(\xi)$, summing as to $j$ from 1 to $n$, and integrating with respect to $\xi$ from $a$ to $b$, we have

$$\frac{K \displaystyle\int_a^b \sum_{j=1}^n z_j(\rho_k, \xi) f_j(\xi)\, d\xi}{\displaystyle\int_a^b \sum_{j=1}^n z_j(\rho_k, \xi) y_j(\rho_k, \xi)\, d\xi} y_s(\rho_k, x) = 1/2\pi i \int_{\gamma_k} \int_a^b \sum_{j=1}^n \rho G_{sj}(x, \xi; \rho^2) f_j(\xi)\, d\xi d\rho.$$

The same relation will hold if the contour integration is about the characteristic value * $-\rho_k$. Hence it follows that

$$(18)\qquad \sum_{k=1}^{m} \frac{\int_a^b \sum_{j=1}^{n} z_j(\rho_k, \xi) f_j(\xi)\, d\xi}{\int_a^b \sum_{j=1}^{n} z_j(\rho_k, \xi) y_j(\rho_k, \xi)\, d\xi}\, y_s(\rho_k, x)$$

$$= 1/2\pi i \int_{C_m} \int_a^b \sum_{j=1}^{n} \rho G_{sj}(x, \xi; \rho^2) f_j(\xi)\, d\xi d\rho,$$

where $C_m$ is a circle with center at the point $\rho = 0$ including just $m$ characteristic values of $\rho$ (namely $\rho_1, \rho_2, \cdots, \rho_m$) which satisfy the relation $-\pi/2 < \arg \rho_k \leqq \pi/2$.

We have thus expressed the sum of $m$ terms of the expansion (7), (8) as a contour integral. If we let $m$ become infinite, we obtain a sequence of circles $C_m$ whose radii become infinite with $m$ and the sum of the series, if it exists, will be given by the limit of the contour integral as $m$ becomes infinite. It should be noted, however, that in the convergence proof it is assumed that the circles $C_m$ are so chosen that no one of them comes closer to a characteristic value of $\rho$ than a distance $D > 0$. This is possible only if we let $m$ become infinite by jumps of $n$ so that sufficiently far out in the $\rho$-plane there are included between two consecutive circles $2n$ characteristic values, namely, the ones which are clustered about the points $k\pi/(b-a)$ and $-k\pi/(b-a)$, $k$ being an integer. The convergence of the series is thus established only when its terms are grouped $n$ at a time. The author has been unable to determine whether or not the series is always convergent when such a grouping is not made.

If the assumptions made concerning the characteristic values, namely, that they are simple and that at each of them $\rho G_{st}(x, \xi; \rho^2)$ has a pole of the first order, do not hold for some one of them, then the proof of the next section, that the limit of the contour integral represents the function $f_s(x)$, is unaltered, but we can no longer say that relation (18) holds. The expansion (7), (8) is thus somewhat less general than the representation of the functions as the limits of contour integrals.

## 5. The Convergence Proof.

In this section we shall establish the main result of this investigation.

Theorem 7. *Let each of the single-valued functions* $f_1(x), f_2(x), \cdots,$

* If $\rho_k$ is a characteristic value, then $-\rho_k$ is also, for $\Delta(\rho) = (-1)^n \Delta(-\rho)$.

*$f_n(x)$ be made up in the interval $a \leqq x \leqq b$ of a finite number of pieces, each piece real, continuous, and with a continuous derivative. Let $\rho_1, \rho_2, \cdots$, denote the sequence of characteristic values of $\rho$ for system (1), (9) such that $-\frac{1}{2}\pi < \arg \rho_k \leqq \frac{1}{2}\pi$; let $y_1(\rho_k, x), y_2(\rho_k, x), \cdots, y_n(\rho_k, x)$ be a set of solutions of system (1), (9) for $\rho = \rho_k$, and let $z_1(\rho_k, x), z_2(\rho_k, x), \cdots, z_n(\rho_k, x)$ be a set of solutions of system (3), (4) for $\rho = \rho_k$. Then if $|\beta, \delta| \neq 0$, if the characteristic values are simple, and if the functions $\rho G_{st}(x, \xi; \rho^2)$ have simple poles at each of the characteristic values, the expansion of $f_s(x)$ connected with the system (1), (9), namely, the infinite series*

$$\sum_{k=1}^{\infty} \frac{\int_a^b \sum_{j=1}^{n} z_j(\rho_k, \xi) f_j(\xi)\, d\xi}{\int_a^b \sum_{j=1}^{n} z_j(\rho_k, \xi) y_j(\rho_k, \xi)\, d\xi}\, y_s(\rho_k, x)$$

*converges for $a < x < b$ to the value $\frac{1}{2}[f_s(x-0) + f_s(x+0)]$; provided that for all large positive integral values of $\mu$ the $n$ terms of the series assso-ciated with the $n$ characteristic values which are clustered about the point $\mu\pi/(b-a)$ are grouped together; for $x = a$ it converges to the value $f_s(a+0)$; for $x = b$ it converges to the value $f_s(b-0)$.*

Consider first the integral

$$1/2\pi i \int_{C_m} \int_a^x \rho G_{ss}(x, \xi; \rho^2) f_s(\xi)\, d\xi d\rho, \qquad a < x < b.$$

Here $\xi$ is less than $x$ so that

$$\rho G_{ss}(x, \xi; \rho^2) = \frac{D_{ss}(\rho, \xi) \cos \rho x + D_{n+s,s}(\rho, \xi) \sin \rho x}{\Delta(\rho)}.$$

Arranging the numerator in descending powers of $\rho$, we have for the leading term $\rho^{2n} |\beta, \delta| \sin^{n-1}\rho(b-a) \cos \rho(\xi - a) \cos \rho(x-b)$, and hence

$$\rho G_{ss}(x, \xi; \rho^2) = \frac{\cos \rho(\xi - a) \cos \rho(x-b)}{\sin \rho(b-a)} + \frac{P_{ss}(x, \rho, \xi)}{\Delta(\rho)},$$

where each term of $P_{ss}(x, \rho, \xi)$ consists of a constant times $\rho^{\mu}$, $0 \leqq \mu \leqq 2n-1$, multiplied by a product of powers of $\sin \rho(b-a)$, $\cos \rho(b-a)$, $\sin \rho(x-b)$, $\cos \rho(x-b)$, $\sin \rho(\xi - x)$, $\sin \rho(\xi - a)$, $\cos \rho(\xi - a)$, the first two of these being the only ones which occur to a power higher than the first, and the total degree being not greater than $n+1$.

An integration by parts with respect to $\xi$ gives

$$(19)\qquad 1/2\pi i \int_{C_m} \int_a^x \frac{\cos\rho(\xi-a)\cos\rho(x-b)}{\sin\rho(b-a)} f_s(\xi)\,d\xi d\rho$$

$$= 1/2\pi i \int_{C_m} 1/\rho \left\{ \left[ \tfrac{1}{2} + \frac{\sin\rho(2x-a-b)}{2\sin\rho(b-a)} \right] f_s(x-0) \right.$$

$$+ \sum_{k=1}^{r_s} \frac{\sin\rho(x_k-a)\cos\rho(x-b)}{\sin\rho(b-a)} [f_s(x_k-0) - f_s(x_k+0)]$$

$$\left. - \int_a^x \frac{\sin\rho(\xi-a)\cos\rho(x-b)}{\sin\rho(b-a)} f'_s(\xi)\,d\xi \right\} d\rho,$$

where $x_1, x_2, \cdots, x_{r_s}$ are the points in the interval $(ax)$ at which $f_s(x)$ has finite discontinuities. As $m$ becomes infinite the second member of this equation approaches as a limit the value $\frac{1}{2}f_s(x-0)$ for

$$1/2\pi i \int_{C_m} \frac{f_s(x-0)}{2\rho}\,d\rho = \tfrac{1}{2}f_s(x-0)$$

and the remaining integrals yield the value zero in the limit.*

If in $P_{ss}(x,\rho,\xi)/\Delta(\rho)$ we divide numerator and denominator by $\rho^{2n}\sin^n\rho(b-a)$, the denominator is bounded away from zero on $C_m$ while the numerator consists of terms similar to the one just considered, except that there occurs in each the factor $1/\rho^\nu$, $\nu \geqq 1$, which is sufficient to carry the integral to zero as $m$ becomes infinite. Consequently we have

$$\lim_{m=\infty} 1/2\pi i \int_{C_m} \int_a^x \rho G_{ss}(x,\xi;\rho^2) f_s(\xi)\,d\xi d\rho = \tfrac{1}{2}f_s(x-0).$$

Likewise one finds that

$$\lim_{m=\infty} 1/2\pi i \int_{C_m} \int_x^b \rho G_{ss}(x,\xi;\rho^2) f_s(\xi)\,d\xi d\rho = \tfrac{1}{2}f_s(x+0),$$

so that

$$\lim_{m=\infty} 1/2\pi i \int_{C_m} \int_a^b \rho G_{ss}(x,\xi;\rho^2) f_s(\xi)\,d\xi d\rho = \tfrac{1}{2}[f_s(x-0) + f_s(x+0)].$$

If $s$ is different from $t$, $\rho G_{st}(x,\xi;\rho^2) = Q_{st}(x,\rho,\xi)/\Delta(\rho)$, where $\rho$ appears in $Q_{st}(x,\rho,\xi)$ to a power not greater than $2n-1$, the coefficients of the various powers of $\rho$ being polynomials of degree not greater than $n$ in $\sin\rho(b-a)$, $\cos\rho(b-a)$, $\sin\rho(x-a)$, $\cos\rho(x-a)$, $\sin\rho(x-b)$, $\cos\rho(x-b)$, $\sin\rho(\xi-a)$, $\cos\rho(\xi-a)$, $\sin\rho(\xi-b)$, $\cos\rho(\xi-b)$, of which $\sin\rho(b-a)$ and $\cos\rho(b-a)$ are the only ones which appear to a power higher than the first. Dividing numerator and denominator by $\rho^{2n}\sin^n\rho(b-a)$ we see as before that each term of the numerator is of the

* See *Bull. Amer. Math. Soc.*, 30 (1924), 413.

form $T/\rho$ where $T$ is bounded on $C_m$ and that the integral of each term yields the value zero in the limit. The desired relation is now established, namely,

$$\lim_{m=\infty} 1/2\pi i \int_{C_m} \int_a^b \sum_{j=1}^{n} \rho G_{sj}(x, \xi; \rho^2) f_j(\xi)\, d\xi d\rho = \tfrac{1}{2}[f_s(x-0) + f_s(x+0)], \quad a < x < b.$$

For $x = b$, equation (19) becomes

$$1/2\pi i \int_{C_m} \int_a^b \frac{\cos \rho(\xi - a)}{\sin \rho(b-a)} f_s(\xi)\, d\xi d\rho = f_s(b-0) + \int_{C_m} \sum_{k=1}^{r_b} \frac{\sin \rho(x_k - a)}{\rho \sin \rho(b-a)} [f_s(x_k - 0) - f_s(x_k + 0)] d\rho - 1/2\pi i \int_{C_m} \int_a^b \frac{\sin \rho(\xi - a)}{\rho \sin \rho(b-a)} f_s'(\xi)\, d\xi d\rho.$$

The terms in the second member of this equation, other than the constant $f_s(b-0)$, yield the value zero as $m$ becomes infinite as do also the integrals of the terms corresponding to $P_{ss}(x, \rho, \xi)/\Delta(\rho)$ and $Q_{st}(x, \rho, \xi)/\Delta(\rho)$. Similarly, if one takes $x = a$ in

$$1/2\pi i \int_{C_m} \int_a^b \rho G_{ss}(x, \xi; \rho^2) f_s(\xi)\, d\xi d\rho,$$

one verifies the part of the theorem concerning the behaviour of the expansion at $x = a$.

## 6. A Generalization of Fourier's Series.

An interesting set of expansions arises if we choose the following boundary conditions:

$$\begin{aligned} &y_1(-\pi) = y_2(\pi), && y'_1(-\pi) = y'_2(\pi), \\ &y_2(-\pi) = y_3(\pi), && y'_2(-\pi) = y'_3(\pi), \\ &\cdots\cdots\cdots && \cdots\cdots\cdots \\ &y_{n-1}(-\pi) = y_n(\pi), && y'_{n-1}(-\pi) = y'_n(\pi), \\ &y_n(-\pi) = y_1(\pi), && y'_n(-\pi) = y'_1(\pi). \end{aligned}$$

These conditions do not satisfy the relation $|\beta, \delta| \neq 0$, so that the expansion problem connected with these conditions and equation (1) requires a separate investigation. If one sets about such an investigation, being guided by the method used in the problem with more general boundary conditions, one finds that the characteristic values are $\mu/n$, $\mu = 0, \pm 1, \pm 2, \pm 3, \cdots$, and that the Green's functions $G_{st}(x, \xi; \rho^2)$ are given by

$$\rho G_{st}(x, \xi; \rho^2) = \frac{\cos \rho[x - \xi + (n - 2s + 2t)\pi]}{2 \sin \rho n \pi}, \qquad s \neq t,$$

$$\rho G_{tt}(x,\xi;\rho^2) = \begin{cases} \dfrac{\cos\rho(x-\xi+n\pi)}{2\sin\rho n\pi}, & -\pi \leqq x < \xi, \\ \dfrac{\cos\rho(x-\xi-n\pi)}{2\sin\rho n\pi}, & \xi < x \leqq \pi. \end{cases}$$

The argument here is so similar to the one for the more general case, although considerably simplified in certain respects, that the reader will have no difficulty in constructing for himself the detailed proof of the following theorem.

THEOREM 8. *Let each of the single-valued functions $f_s(x)$, $s = 1, 2, \cdots, n$, be made up in the interval $-\pi \leqq x \leqq \pi$ of a finite number of pieces, each piece real, continuous, and with a continuous derivative. The infinite series*

$$\tfrac{1}{2}a_0 + \sum_{k=1}^{\infty} \{a_k \cos \frac{k[x+(n-2s+1)\pi]}{n} + b_k \sin \frac{k[x+(n-2s+1)\pi]}{n},$$

*where*

$$a_k = 1/n\pi \int_{-\pi}^{\pi} \sum_{j=1}^{n} f_j(\xi) \cos \frac{k[\xi+(n-2j+1)\pi]}{n}\, d\xi,$$

$$b_k = 1/n\pi \int_{-\pi}^{\pi} \sum_{j=1}^{n} f_j(\xi) \sin \frac{k[\xi+(n-2j+1)\pi]}{n}\, d\xi,$$

*converges for $-\pi < x < \pi$ to the value $\frac{1}{2}[f_s(x+0)+f_s(x-0)]$. For $x = -\pi$, it converges to $\frac{1}{2}[f_s(-\pi+0)+f_{s+1}(\pi-0)]$, where, for $s = n$, $f_{s+1}(\pi-0)$ is to be replaced by $f_1(\pi-0)$. For $x = \pi$, it converges to $\frac{1}{2}[f_{s-1}(-\pi+0)+f_s(\pi-0)]$, where, for $s = 1$, $f_{s-1}(-\pi+0)$ is to be replaced by $f_n(-\pi+0)$.*

For $n = 1$ the theorem merely asserts the convergence of Fourier's series.

## 7. Simultaneous Expansions of Functions of $\nu$ Variables.

When a problem of the expansion of functions of a single variable is solved by Birkhoff's method, there always follows as an immediate corollary a theorem concerning the expansion of functions of several variables. In the present case let us consider the system

$$(20) \qquad d^2y_{st}/dx^2{}_t + \rho_t{}^2 y_{st} = 0, \qquad Y_{\sigma t} = 0,$$
$$s = 1, 2, \cdots, n, \qquad \sigma = 1, 2, \cdots, 2n,$$

and its adjoint, there being one such system for each value of $t$ from 1 to $\nu$. We suppose that the conditions $Y_{1t} = 0$, $Y_{2t} = 0, \cdots, Y_{2n,t} = 0$, are linear homogeneous expressions in the $4n$ quantities $y_{st}(a_t)$, $y'_{st}(a_t)$, $y_{st}(b_t)$, $y'_{st}(b_t)$, $s = 1, 2, \cdots, n$, and that the determinant corresponding to $|\beta, \delta|$ does not vanish. If we denote the characteristic values of $\rho_t$ for which

$-\frac{1}{2}\pi < \arg \rho_t \leqq \frac{1}{2}\pi$ by $\rho_{t1}, \rho_{t2}, \cdots$, and write $\eta_s(x)$ and $\zeta_s(x)$ for $\prod_{t=1}^{\nu} y_{st}(\rho_{tk_t}, x_t)$ and $\prod_{t=1}^{\nu} z_{st}(\rho_{tk_t}, x_t)$ respectively, where $y_{st}(\rho_{tk_t}, x_t)$ and $z_{st}(\rho_{tk_t}, x_t)$, $s = 1, 2, \cdots, n$, are sets of solutions of (20) and its adjoint respectively for $\rho_t = \rho_{tk_t}$, then the coefficients in the formal expansions

$$f_s(x_1, x_2, \cdots, x_\nu) = \sum_{k_1=1}^{\infty} \sum_{k_2=1}^{\infty} \cdots \sum_{k_\nu=1}^{\infty} c_{k_1 k_2 \cdots k_\nu}\, \eta_s(x), \qquad s = 1, 2, \cdots, n, \tag{21}$$

are readily found to be

$$c_{k_1 k_2 \cdots k_\nu} = \frac{\int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_\nu}^{b_\nu} \sum_{j=1}^{n} f_j(\xi_1, \xi_2, \cdots, \xi_\nu)\, \zeta_j(\xi)\, d\xi_\nu \cdots d\xi_2 d\xi_1}{\int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_\nu}^{b_\nu} \sum_{j=1}^{n} \eta_j(\xi)\zeta_j(\xi)\, d\xi_\nu \cdots d\xi_2 d\xi_1}.$$

If $G_{sjt}(x_t, \xi_t; \rho_t^2)$, $s, j = 1, 2, \cdots, n$, denote the Green's functions of (20), one easily obtains that the sum of the series for $f_s(x_1, x_2, \cdots, x_\nu)$ is given by the following limit:

$$\lim_{r_1, r_2, \cdots r_\nu = \infty} 1/(2\pi i)^\nu \int_{C_1} \int_{C_2} \cdots \int_{C_\nu} \int_{a_1}^{b_1} \int_{a_2}^{b_2} \cdots \int_{a_\nu}^{b_\nu} \sum_{j=1}^{n} f_j(\xi_1, \xi_2, \cdots, \xi_\nu) \prod_{t=1}^{\nu} \rho_t G_{sjt}(x_t, \xi_t; \rho_t^2)\, d\xi_\nu \cdots d\xi_2 d\xi_1\, d\rho_\nu \cdots d\rho_2 d\rho_1,$$

where $C_t$ is a circle in the $\rho_t$-plane with center at the origin whose radius $r_t$ becomes infinite in such a way that $C_t$ does not come nearer to any characteristic value of $\rho_t$ than a distance $D > 0$. If we restrict the variables $x_j$ by the inequalities $a_j \leqq x_j \leqq b_j$, $j = 1, 2, \cdots, \nu$, and suppose that $f_s(x_1, x_2, \cdots, x_\nu)$, considered as a function of each variable separately, satisfies in these intervals the conditions imposed on $f_s(x)$ in Theorem 7, and if we suppose that the characteristic values are simple in every case and that the functions $\rho_t G_{sjt}(x_t, \xi_t; \rho_t^2)$ have first order poles at $\rho_t = \rho_{t1}, \rho_{t2}, \cdots$, we see that the expansion of $f_s(x_1, x_2, \cdots, x_\nu)$ converges to the value

$$1/2^\nu \sum f_s(x_1 \pm 0, x_2 \pm 0, \cdots, x_\nu \pm 0), \qquad a_j < x_j < b_j, \tag{22}$$

where the summation sign indicates that one takes the sum of the $2^\nu$ terms obtained by choosing all possible combinations of the positive and negative signs, provided that, before the summation as to each $k_j$ in (21) is carried out, the terms of the series are grouped as they were in Theorem 7. If $x_j = a_j$ (or $b_j$) $x_j \pm 0$ is to be replaced by $a_j + 0$ (or $b_j - 0$) in (22) and the summation is to be multiplied by 2.

It is evident that Theorem 8 may be generalized in a similar manner.

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# The Distribution of Lift Over Thin Wing Sections.

By C. A. Shook.

## *Introduction.*

In the application of theoretical Hydrodynamics to practical problems, two assumptions are usually made, in order to simplify the treatment. First, viscosity is neglected. While all fluids show a certain amount of internal friction, yet in the case of water and air these forces are small when compared with the other forces which enter, as for example the forces due to inertia. Second, the fluid is assumed to be incompressible. This assumption leads to a small but appreciable error in the case of air. The approximation is much better in the case of water.

Under these assumptions we propose to find the distribution of lift over thin wing sections, the thin wing being treated mathematically as a curve. A possible procedure is to make use of the known flow about a circle, (described in Sec. 2), and to seek a transformation of the circle into the desired section. We have used a form of transformation given by Munk.* There result two infinite series which, on account of slow convergence, are not well suited to numerical calculation. A brief indication of the method followed with the results is given in Part III. The general method of procedure here may be said to have originated with Prandtl.† An important contribution was made by Kutta,‡ who introduced the idea of a circulatory flow just sufficient to make the fluid leave the rear edge of the wing smoothly. Joukowski § has shown how certain wing-like profiles may be obtained by the transformation of a circle.

A more useful method is that developed in Part II. Here I assume a certain vertical component of velocity along a horizontal line segment, in addition to the ordinary horizontal flow, and find the horizontal component

* Munk, "General Theory of Thin Wing Sections," Report 142, National Advisory Committee for Aeronautics.

† A summary is given by Prandtl in Report 116, *Ibid.*

‡ Kutta, "Auftriebskräfte in strömenden Flüssigkeiten," *Illustr. aeronaut. Mitteilungen*, 1902, S. 133. See also, *Proc. Bavarian Acad. of Sc., Math.-Phys. Class*, 1910 and 1911.

§ "Über die Konturen der Tragflächen der Drachenflieger," *Zeitschrift für Flugtechnik und Motorluftschiffahrt*, 1910, S. 281.

corresponding to it, thus obtaining the total velocity. This assumption of a vertical velocity has the effect of reducing the problem to a well-known one in potential theory, namely, the problem of finding a potential function when its normal derivative is given over a certain contour. Here the results are in terms of integrals of a general character. These integrals, in the special case treated later, lend themselves readily to numerical calculation.

Munk * has recently obtained results similar to parts of Part II of this paper, by considering a whole series of two dimensional flows, each of which, he proves, yields a pure vertical velocity component at one point of the straight line, and a pure horizontal component at all other points. Finally he shows how to compute the horizontal velocity at a point of the line for the sum of these flows. In a résumé of Munk's work, Ames † has reproduced this analysis.

For a detailed treatment of uniplanar flow, the fundamental facts concerning which are briefly mentioned in Part I, reference may be made to Appell, *Traité de Mécanique Rationnelle,* Tome 3 and Lamb, *Hydrodynamics.* Part IV contains several numerical tables calculated from the results of Part II. An indication of the method of calculation is also given. Following is an outline:

I. UNIPLANAR FLOW.

1. Uniplanar Flow in general.
2. Flow about Circle and Line.
3. Lift and Moment Coefficients for Straight Line.

II. GENERAL SOLUTION IN TERMS OF DEFINITE INTEGRALS.

4. Lift and Moment Coefficients for Thin Wing Section.
5. Application of General Solution to Tail Plane.

III. SOLUTION IN SERIES.

6. Introduction; the Series Transformation.
7. Density of Pressure; Hinge Force.
8. Hinge Moment of Elevator Force.

IV. NUMERICAL TABLES.

9. Method of Computation.

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* Report 191 (1924), N. A. C. A., pp. 6-10.
† Report 213 (1925), N. A. C. A., pp. 23-26.

## I. UNIPLANAR FLOW IN GENERAL.

1. *Uniplanar Flow.*—Consider a two-dimensional flow about a closed curve $C$, situated in the plane of the complex variable $\zeta = \xi + i\eta = r\,e^{i\theta}$. If we assume that the flow is irrotational and stationary, and that the velocity at infinity is prescribed, then there exists the *complex potential function,*

$$f(\zeta) = \phi(\xi, \eta) + i\,\psi(\xi, \eta)$$

such that for any point, the component velocities along the coordinate axes are

$$u = \partial\phi/\partial\xi, \qquad v = \partial\phi/\partial\eta \tag{1·1}$$

$\phi$ is called the *velocity potential.* $f'(\zeta)$, the derivative of $f(\zeta)$, has the properties,

$$\text{Real part of } f'(\zeta) = u$$
$$\text{Imaginary part of } f'(\zeta) = -v$$
$$|f'(\zeta)| = \sqrt{u^2 + v^2}$$

Let the $\zeta$ plane be transformed according to the equation $\zeta = g(z)$, where $z = x + iy$, sending $C$ into $C'$ and such that to $\zeta = \infty$ corresponds $z = \infty$. Then $f[g(z)]$ will be the potential function of the flow about $C'$. Also, if $w$ and $w'$ resp. are the magnitudes of the velocities at corresponding points in the $\zeta$ and $z$ planes, then we have,

$$w' = |d\zeta/dz|\,w \tag{1·2}$$

2. *Flow about Circle and Line.*—If at the point at infinity,

$$u = V\cos a, \qquad v = V\sin a,$$

then the flow about the circle is given by,

$$f_1(\zeta) = V\left(e^{-ia}\,\zeta + \frac{1}{e^{-ia}\,\zeta}\right) \tag{2·1}$$

whence,

$$\phi_1 = V(r + 1/r)\cos(\theta - a) \tag{2·2}$$

If to this we add the circulatory flow, with circulation $\Gamma$ in the clockwise direction, for which,

$$f_2(\zeta) = i\Gamma/2\pi \log \zeta, \text{ whence, } \phi_2 = -\Gamma\theta/2\pi$$

we get for the total velocity potential,

$$\phi = V(r + 1/r)\cos(\theta - a) - \Gamma\theta/2\pi \tag{2·3}$$

The value of $\Gamma$ designed to make the velocity vanish at $\zeta = 1$ is

$$\Gamma = 4\pi V \sin \alpha \tag{2·4}$$

so that the velocity at points of the circle is,

$$-2V[\sin(\theta - \alpha) + \sin \alpha].$$

The transformation,

$$z = \zeta + 1/\zeta \tag{2·5}$$

maps the unit circle in the $\zeta$ plane into the line connecting $\pm 2$ in the $z$ plane. For points of the circle,

$$|\, d\zeta/dz \,| = \frac{1}{2 \sin \theta} \tag{2·6}$$

Hence the velocity at points of the line is,

$$w = -V[\sin(\theta - \alpha) + \sin \alpha] \frac{1}{\sin \theta} \tag{2·7}$$

where $x = 2 \cos \theta$.

3. *Lift and Moment Coefficients for Straight Line.*—The dynamic pressure $p$ at any point of the straight line is found by Bernouilli's Equation,

$$p = -\rho/2 \; w^2,$$

where $\rho$ is the density of the fluid. The lift is now found by integrating $p$. For the portion of the straight line to the right of $x_1 = 2 \cos \theta_1$, the lift is,

$$4 q_0 \sin 2\alpha \; (\theta_1 - \sin \theta_1)$$

where, $q_0 = \rho/2 \; V^2$. The moment about the center is found by integrating $p \cdot x$ and is

$$8 q_0 \sin 2\alpha \left(\sin \theta_1 - \frac{\theta_1}{2} - \frac{\sin 2\theta_1}{4}\right).$$

The *absolute lift coefficient* is defined as the lift divided by $q_0 t$, where $t$ is the chord of the wing, in this case 4. For the portion of the wing and the whole wing I denote this coefficient by $A_1 a$ and $A_2 a$ resp. The *absolute moment coefficient* is defined as the moment divided by $q_0 t^2$. The corresponding moment coefficients I call $A_3 a$ and $A_4 a$.

The following values follow from the results given above,

$$\left.\begin{aligned} A_1 &= 2(\theta_1 - \sin\theta_1) \\ A_2 &= 2\pi \\ A_3 &= \sin\theta_1 - \tfrac{1}{2}\theta_1 - \tfrac{1}{4}\sin 2\theta_1 \\ A_4 &= -\pi/2. \end{aligned}\right\} \qquad (3{\cdot}1)$$

the arc $2a$ being assumed small enough to be identified with its sine.

## II. GENERAL SOLUTION IN TERMS OF DEFINITE INTEGRALS.

4. *Lift and Moment Coefficients for Thin Wing Sections.*—We now attack the problem of finding the aerodynamic forces on thin wing sections which differ not greatly from a straight line, that is, whose curvature is small. Suppose that we have a curve connecting the points $-2$ and $+2$ in the $z$ plane. The ordinate of this curve is denoted by $s$, so that its equation is $s = s(x)$, $x$ being the abscissa, and $s(\pm 2) = 0$. We assume that $s$ remains small and the $D_x s$ exists and remains small. $s(x)$ is to be continuous in the interval $-2 \leqq x \leqq +2$, while $D_x s$, in the same interval, is to be continuous except at a finite number of points, at each of which it may have a finite discontinuity.

We are now led to the following assumption with regard to the vertical component $v$ of the velocity along the straight line:

$$v = V D_x s \qquad (4{\cdot}1)$$

where $V$ denotes the velocity of the air at a remote distance. In fact, the actual velocity along the $s$-curve will differ from the velocity along the straight line by a small quantity $\delta$, so that this velocity is $V + \delta$. The vertical component of this velocity is found by multiplying the magnitude of the velocity by $\sin\tau$, where $\tau$ is the inclination of the $s$-curve to the $x$-axis. But $\sin\tau$ may be replaced by $\tan\tau$, to the first order of approximation. And thus we arrive at the value of $v$ given in (4·1).

We now wish to find the horizontal component $u$ of this same velocity and this will be the additional velocity, over $V$, caused by the change of shape from the straight line to the $s$-curve.

We seek the potential function of the flow which is characterised by a vertical velocity component $v$ along the line joining $-2$ and $+2$. We start with a unit circle, the section of a cylinder, located in the $\zeta$ plane, with center at the origin. Suppose that at the point $\zeta_0 = e^{i\theta_0}$ there issues a source of

strength $v$, while at the point $\bar{\zeta}_0 = e^{-i\theta_0}$ there is located a sink of equal strength, namely $-v$. The potential function of this flow is,

$$F(\zeta) = v/\pi \log \frac{\zeta - \zeta_0}{\zeta - \bar{\zeta}_0} \tag{4·2}$$

The unit circle is easily seen to be a stream line of this flow. For, from Fig. 1 we have, if $\zeta$ is on the unit circle,

$$\zeta - \zeta_0 = r_1 e^{i\theta_1}$$

and,

$$\zeta - \bar{\zeta}_0 = r_2 e^{i\theta_2}$$

so that,

$$F(\zeta) = v/\pi \log \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}}$$

$$= v/\pi \log r_1/r_2 + i\, v/\pi\, (\theta_1 - \theta_2).$$

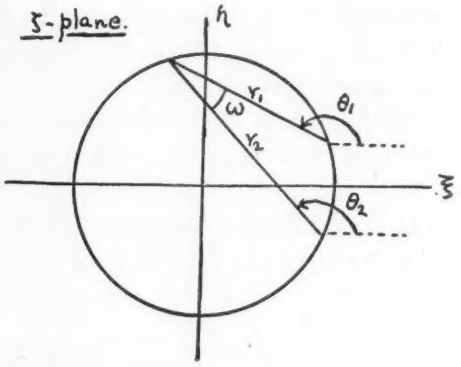


FIG. 1.

Since $\theta_1 - \theta_2 = \omega = \text{const.}$, when $\zeta$ is on the circle, it results that the imaginary part of the potential function is constant over the circle and hence that the circle is a stream line. We can now imagine that this circular stream line is replaced by a cylindrical barrier and there is no flow inside the cylinder. The flow outside will be unaltered. This is analogous to replacing an equipotential surface by a conductor in an electrical problem.

It is now easy to find the velocity along the circle arising from this flow. The velocity function is,

$$F'(\zeta) = v/\pi \left[\frac{1}{\zeta - \zeta_0} - \frac{1}{\zeta - \bar{\zeta}_0}\right]$$

$$= + \frac{v}{\pi} \frac{i \sin\theta_0\, e^{-i\theta}}{\cos\theta - \cos\theta_0}, \text{ for points on the circle.}$$

The actual velocity is the absolute value of $F'(\zeta)$, or,

$$\pm \frac{v}{\pi} \frac{\sin\theta_0\, e^{-i\theta}}{\cos\theta - \cos\theta_0} \text{ according as } \theta \gtrless \theta_0.$$

We now make a transformation of the flow according to the equation,

$$z = \zeta + 1/\zeta \tag{4·4}$$

We have seen that this transformation maps the unit circle into the straight line connecting $+2$ and $-2$ and that $x = 2\cos\theta$. Since,

$$|\, d\zeta/dz \,| = \frac{1}{2\sin\theta},$$

we have for the horizontal velocity along the straight line, due to the given flow,

$$\frac{v}{2\pi} \frac{1}{\sin\theta} \frac{\sin\theta_0}{\cos\theta - \cos\theta_0}$$

or,

$$\pm \frac{v}{2\pi} \frac{2}{x - x_0} \frac{\sqrt{4 - x_0^2}}{\sqrt{4 - x^2}}$$

This is the velocity at the point $x$ due to the elementary flow from the point $x_0$. The upper and lower signs refer respectively to the top and bottom of the line. The insertion of the proper signs is necessary, since the method I have used yields only the magnitude of the velocity. The choice of signs results from a consideration of the flow, which is indicated in Fig. 2.

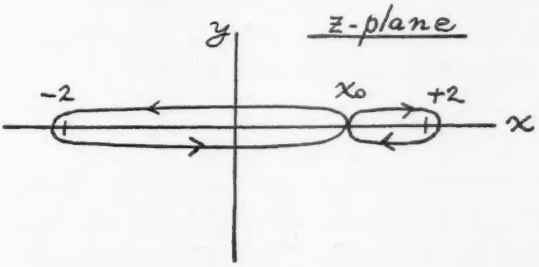


FIG. 2.

An integration extending over the line now obviously gives the total horizontal component $u$ due to all such elementary flows. Thus,

$$u = \pm V/2\pi \int_{-2}^{+2} \frac{ds}{dx} \frac{2}{x - x_0} \frac{\sqrt{4 - x_0^2}}{\sqrt{4 - x^2}}\, dx_0 = \pm V/2\pi\, J, \text{ say} \tag{4·5}$$

From Bernouilli's equation we get, $dp = -\rho\, w\, dw$, where $dw = u$, $w = V$ approximately, and $dp$ is the increase in pressure. Hence,

$$\begin{aligned} dp &= -\rho\, V\, u = \mp \rho\, V\ V/2\pi\ J \\ &= \mp\ q_0/\pi\ J \end{aligned} \tag{4·6}$$

Assuming for the moment that $J$ is positive, (4·6) tells us the pressure on the top is decreased and the pressure on the bottom is increased, the change in each case being the same. Hence taking both sides of the line into account, the increase in *lift* at the point $x$ is,

$$dp = 2q_0/\pi\ J \tag{4·7}$$

The lift on any part of the wing may now be found by an integration. The absolute lift coefficient for this flow is found by dividing the actual lift by $4q_0$, since the total length of our wing is 4. It is a little more convenient to change the limits in the expression for $J$. Also, let $x_2$ be any value between $-1$ and $+1$. Then we find finally for the lift coefficient for this flow, corresponding to the lift of the wing to the right of $x_2$, the following integral,

$$C_3 = 2/\pi \int_{x_2}^{1} dx \int_{-1}^{1} \frac{ds}{dx}\ \frac{1}{x - x_0}\ \frac{\sqrt{1 - x_0^2}}{\sqrt{1 - x^2}}\ dx_0 \tag{4·8}$$

Physical considerations demand that the flow should leave the section at the point $+1$. In order then that the rear rest point of our section should be $+1$, it is necessary to superimpose a circulation flow. The potential function of the circulatory flow around the circle is $i\Gamma \log \zeta$ and $\Gamma$ is the actual velocity along the circle (since the radius is unity), in the direction of $\theta$ decreasing. It results therefore that we must have

$$\begin{aligned} \Gamma &= -\ \frac{v}{\pi}\ \frac{\sin\ \theta_0}{1 - \cos\theta_0} \\ &= -\ \frac{V}{\pi}\ \frac{ds}{dx}\ \frac{\sqrt{2 + x_0}}{\sqrt{2 - x_0}} \end{aligned}$$

The total circulation resulting from all these elementary flows is then given by the integral,

$$\Gamma = -\ \frac{V}{\pi} \int_{-2}^{+2} \frac{ds}{dx}\ \frac{\sqrt{2 + x_0}}{\sqrt{2 - x_0}}\ dx_0 \tag{4.9}$$

To get the corresponding horizontal velocity along the top of the section, we have to divide by $2 \sin \theta$ or $\sqrt{4 - x^2}$. The additional pressure on the

top side will be $-\rho V$ times this additional velocity. This will have to be multiplied by $-2$ as before, to get the entire change in lift at a point of the section. Performing these operations we obtain for the additional lift at a point $x$ of the section, due to the circulation flow,

$$dp = \frac{-2\rho V^2}{\sqrt{4-x^2}} \frac{1}{\pi} \int_{-2}^{+2} \frac{ds}{dx} \frac{\sqrt{2+x_0}}{\sqrt{2-x_0}} dx_0 \qquad (4{\cdot}10)$$

The lift exerted on the wing to the right of the point $x = x'_2 = 2x_2$ is now found by integrating with respect to $x$ between the proper limits. This gives for the corresponding lift,

$$L = -\rho V^2 \, 2/\pi \int_{x'_2}^{+2} dx \int_{-2}^{+2} \frac{ds}{dx} \frac{1}{\sqrt{4-x^2}} \frac{\sqrt{2+x_0}}{\sqrt{2-x_0}} dx_0 \qquad (4{\cdot}11)$$

To get the absolute coefficient we have to divide by $4q_0$. This yields, on changing the limits of integration suitably,

$$C_4 = -2/\pi \int_{x_2}^{+1} dx \int_{-1}^{+1} \frac{ds}{dx} \frac{1}{\sqrt{1-x^2}} \frac{\sqrt{1+x_0}}{\sqrt{1-x_0}} dx_0 \qquad (4{\cdot}12)$$

To obtain the moments about the center, we have merely to insert the factor $x$ under the integral signs in the expressions for the two lifts. This becomes $2x$ on the change of variable, thus introducing the factor 2. To pass then to the absolute coefficient of the moment we have to divide by $q_0 t^2$ instead of by $q_0 t$; that is, we have to divide by $16q_0$ instead of by $4q_0$. Thus there result the following two absolute moment coefficients,

$$C_5 = 1/\pi \int_{x_2}^{+1} dx \int_{-1}^{+1} \frac{ds}{dx} \frac{x}{x-x_0} \frac{\sqrt{1-x_0^2}}{\sqrt{1-x^2}} dx_0 \qquad (4{\cdot}13)$$

$$C_6 = -1/\pi \int_{x_2}^{+1} dx \int_{-1}^{+1} \frac{ds}{dx} \frac{x}{\sqrt{1-x^2}} \frac{\sqrt{1+x_0}}{\sqrt{1-x_0}} dx_6 \qquad (4.14)$$

$C_5$ and $C_6$ must be added to obtain the additional absolute moment coefficient arising from the change from the straight line to the $s$-curve.

It is to be noted that no use has been made of the fact that the $s$-curve goes through the point $(-1, 0)$, although it must pass through the point $(+1, 0)$, since this was taken as the rear rest point. The section might for example be defined by the equation,

$$s = c/4 \, [3 - 2x - x^2],$$

where $c$ is the ordinate at $x = -1$.

The coefficients just deduced must now be added to those found in Sec. 3 to obtain the total coefficients. Let $C_l$ denote the total absolute lift coefficient corresponding to those forces acting to the right of the point $x_2$ and $C_{m_0}$ the total absolute coefficient of the moment of these same forces about the center. Then we shall have,

$$C_l = A_1\alpha + C_3 + C_4 \tag{4·15}$$

$$C_{m_0} = A_3\alpha + C_5 + C_6 \tag{4·16}$$

These are the two quantities which we set out to find. The coefficients corresponding to the forces acting on the whole wing are now easily found by putting $x_2 = -1$ or $\theta_1 = \pi$, where $\cos\theta_2 = x_2$. The quantities $C_l$ and $C_{m_0}$ apply to a thin wing of general shape. The effect of the shape of the wing, which is described by the function giving the $s$-curve, is represented by the appearance in the four integrals of the quantity $D_x s$. The moment about any point of the wing can be found from the lift and the moment about the center.

5. *Application of General Solution to Tail Plane.*—These formulas may now be applied to the case of the tail plane, which has for its section a broken line consisting of two segments hinged at their intersection. The forward plane or stabilizer is rigidly attached to the airplane while the rear plane or elevator can be rotated about the hinge, thereby altering the angle between stabilizer and elevator. This angle is denoted by $\beta$. Making use of the fact that the leading rest point need not coincide with $(-1, 0)$, I assume that the stabilizer is parallel to the $x$ axis and that the elevator passes through $(+1, 0)$. The abscissa of the hinge we call $x_1$ and write, $x_1 = \cos\theta_1$. The following will then hold for this case,

$$\begin{aligned} ds/dx &= 0, && -1 \leqq x < x_1 \\ &= -\beta, && x_1 < x \leqq +1 \end{aligned}$$

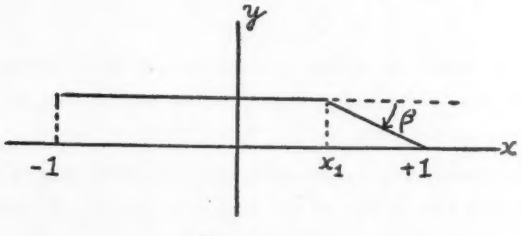


FIG. 4.

$C_3$ to $C_6$ may now be evaluated for this case. Henceforward $C_3$ to $C_6$ shall stand for their values in this particular case. The following values are obtained

$$C_3 = -2\beta/\pi \left[\theta_1 \sin\theta_2 + (\cos\theta_2 - \cos\theta_1) \log \left| \frac{\tan\theta_2/2 + \tan\theta_1/2}{\tan\theta_2/2 - \tan\theta_1/2} \right| \right] \quad (5.1)$$

$$C_4 = 2\beta/\pi \left[\theta_1\theta_2 + \theta_2 \sin\theta_1\right] \quad (5{\cdot}2)$$

$$C_5 = -\beta/2\pi \left[\theta_1\theta_2 + \sin\theta_1 \sin\theta_2 + \theta_1 \sin\theta_2 \cos\theta_2 - \theta_2 \sin\theta_1 \cos\theta_1 + (\sin^2\theta_1 - \sin^2\theta_2) \log \left| \frac{\tan\theta_2/2 + \tan\theta_1/2}{\tan\theta_2/2 - \tan\theta_1/2} \right| \right] \quad (5{\cdot}3)$$

$$C_6 = \beta/\pi \sin\theta_2 \left[\theta_1 + \sin\theta_1\right] \quad (5{\cdot}4)$$

If we are dealing with the forces on the elevator alone, we have to put $\theta_2 = \theta_1$, and thus we get the simpler expressions,

$$C_3 = -2\beta/\pi \ \theta_1 \sin\theta_1 \quad (5{\cdot}5)$$

$$C_4 = 2\beta/\pi \ [\theta_1^2 + \theta_1 \sin\theta_1] \quad (5{\cdot}6)$$

$$C_5 = -\beta/2\pi \ [\theta_1^2 + \sin^2\theta_1] \quad (5.7)$$

$$C_6 = \beta/\pi \ \sin\theta_1 \ [\theta_1 + \sin\theta_1] \quad (5{\cdot}8)$$

To obtain the coefficients corresponding to the forces acting over the whole wing, we have to put $\theta_2 = \pi$ and get

$$C_3 = 0 \quad (5{\cdot}9)$$

$$C_4 = 2\beta[\theta_1 + \sin\theta_1] \quad (5{\cdot}10)$$

$$C_5 = -\beta/2 \ [\theta_1 - \sin\theta_1 \cos\theta_1] \quad (5{\cdot}11)$$

$$C_6 = 0 \quad (5{\cdot}12)$$

The vanishing of $C_3$ for the whole wing might have been predicted. For the entire lift of a wing due to a straight flow is zero. This was pointed out in the case of a circular section by d'Alembert and the fact is usually referred to as d'Alembert's paradox. It is only comparatively recently that the lift of an airplane wing has been ascribed to the circulatory flow. This flow is set up by the formation of vortices, caused by the flow at a very great (theoretically infinite) velocity at the trailing edge. These vortices continue to be formed until the circulation is such that the air leaves the training edge smoothly. Since a vortex always occupies the same fluid particles, these vortices are left behind and the flow is practically stationary until a change in the angle of attack causes fresh vortices to be given off, thereby altering the circulatory flow.

If in equations (5·9)-(5·12) we put $\theta_1 = \pi$, then the wing consists of only the elevator, that is, it becomes a straight line section. The lift and moment coefficients become respectively,

$$2\pi\beta \qquad \text{and} \quad -\pi/2 \ \beta$$

and these, it will be seen, correspond with those already found for a straight line in Sec. 3.

The force and moment coefficients which have been deduced are estimated per unit span of the theoretically infinite wing. Near the outer edges of a finite wing the theory ceases to apply. In practice the finite span is taken into consideration as follows; the angle of attack is imagined to be decreased by the amount

$$\frac{L}{\pi\ b^2\ q_0}$$

where $L =$ total lift, $b =$ span and $q_0 = \frac{1}{2}\rho\ V^2$.

## III. SOLUTION IN SERIES.

6. *Introduction; the Series Transformation.*—In this part we develop the solution of the tail plane problem by means of a transformation of the complex plane in the form of an infinite series. The method will be indicated rather than the details. A great deal of computation was necessary in this particular case since the resulting series converged slowly and it was necessary most of the time to take into account as many as twenty terms of the series. The present method however may prove to be somewhat shorter than that previously developed in Part II in certain cases awaiting solution in which it may happen that the integrals given in equations (4·8), (4·12), (4·13) and (4·14) can not readily be evaluated in finite form. It is the hope that the method may prove useful in other cases rather than its usefulness in solving the problem in hand which has prompted us to append a brief indication of it.

Start with the assumptions and notation of the first paragraph of Sec. 4. We will imagine that by means of the transformation (2·5), the $s$-curve came from a curve in the $\zeta$ plane which we may call the $\sigma$-curve, and $\sigma$ represents the small distance, measured positively away from the origin along a radius, from the unit circle to the $\sigma$-curve. Calling $|\ s\ | = |\ dz\ |$ and $|\ \sigma\ | = |\ d\zeta\ |$ we have

$$\sigma = \frac{1}{2 \sin \theta}\ s \qquad (6{\cdot}1)$$

The transformation which maps the unit circle into the $\sigma$-curve is imagined performed by the relation:

$$1/\zeta_1 = 1/\zeta\ (1 + a_1/\zeta + a_2/\zeta^2 + \cdot\ \cdot\ \cdot\ \cdot\ \cdot) \qquad (6{\cdot}2)$$

where $\zeta_1$ is the variable in the plane of the $\sigma$-curve. If we choose $a_n =$

$-A_n - iB_n$, where $A_n$ and $B_n$ are the coefficients of $\cos n\theta$ and $\sin n\theta$ respectively, in the expansion of $\sigma$ in a Fourier Series, we get:

$$1/\zeta_1 = 1/e^{i\theta}\,[1 - \sigma - i\epsilon] \tag{6·3}$$

where $\epsilon = \sum_n B_n \cos n\theta$ and we have put $r = 1$. The $A_n$ are all zero since $\sigma$ is an odd function of $\theta$ as is shown by (6·1) when we remember that $s$ is positive. Since $\sigma$ and $\epsilon$ are small,

$$\zeta_1 = e^{i\theta}(1 + \sigma) + e^{i\theta} \cdot i \cdot \epsilon\,. \tag{6·4}$$

But in order that $\zeta_1$ should describe the $\sigma$-curve when $\zeta$ describes the unit circle, we ought to have,

$$\zeta_1 = e^{i\theta}\,(1 + \sigma),$$

and the above transformation is thus not exact. We have the additional vector, $e^{i\theta} \cdot i \cdot \epsilon$. the absolute value of this vector is $\epsilon$ while its argument is $\theta + \pi/2$. Thus the vector is tangent to the circle, hence approximately to the $\sigma$-curve, and therefore the end of the vector is situated on the $\sigma$-curve approximately. Therefore (6·3) maps the circle into the $\sigma$-curve and in the transformation the angle $\theta$ is increased in amount equal to $\epsilon$.

7. *Density of Pressure; Hinge Force.*—Let the $s$-curve consist of two line segments through $+2$ and $-2$, having intercepts $y_1$ and $y_2$ resp. on the imaginary axis. These segments are imagined hinged at their intersection, whose abscissa we call $x_1$ and write $x_1 = 2\cos\theta_1$. The hinge force will be equal to the lift experienced by that part of the wing to the right of the hinge. The quantity is obtained by integrating the pressure, the limits being chosen as before to cover both the top and the bottom of the portion of the wing under consideration.

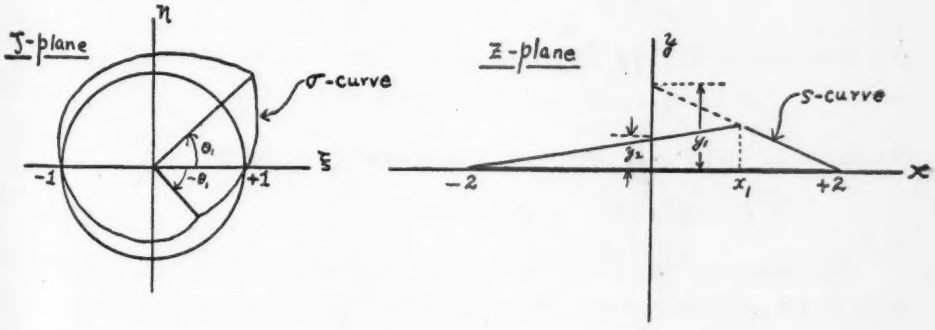


FIG. 4.

By (6·3) the circle is transformed into the $\sigma$-curve. Since, when $\theta = 0$ or $\pi$, $\sigma = 0$, the $\sigma$-curve goes through $+1$ and $-1$. But $+1$ is the rear rest point of the $\sigma$-curve, since that point corresponds by (2·5) to $+2$ of the $s$-curve. Therefore the flow around the circle must be chosen so that it will leave the circle at the point which corresponds to $+1$ of the $\sigma$-curve. Now $\epsilon = \sum_n B_n \cos n\theta$. Let $\epsilon_0$ represent the point of the $\sigma$-curve into which $\theta = 0$ of the circle is transformed. Then $\epsilon_0 = \sum_n B_n$. Since $\epsilon$ is small and continuous, $\theta = -\epsilon_0$ is also approximately the point of the circle which is transformed into $+1$ of the $\sigma$-curve. Thus the flow around the circle must leave it at $\theta = -\epsilon_0$. Now when the angle of attack is $a$ and there is no circulation, the velocity of flow around the circle is $-2V\sin(\theta - a)$ and this becomes $-2V(\sin\theta - a\cos\theta)$, where now the angle $a$ is to be small. To this we must add the velocity due to circulation, $-2V(a + \epsilon_0)$. This gives then for the total velocity around the circle,

$$-2V[\sin\theta - a\cos\theta + (a + \epsilon_0)] \tag{7·1}$$

When $\theta = -\epsilon_0$ this velocity vanishes.

The velocity at corresponding points of the straight line is found by multiplying this by,

$$|\, d\zeta/dz \,| \quad \text{or} \quad \frac{1}{2\sin\theta}$$

and this gives,

$$-V\left[1 - a\cot\theta + \frac{(a + \epsilon_0)}{\sin\theta}\right]$$

The pressure at any point is thus seen to be,

$$p = -q_0\left[1 - 2a\cot\theta + \frac{2(a + \epsilon_0)}{\sin\theta}\right]. \tag{7·2}$$

An integration now gives the following value for the total pressure:

$$P_1 = 8q_0\left[a\,(\theta_1 - \sin\theta_1) + \epsilon_0\theta_1\right] \tag{7·3}$$

Remembering that $t = 4$, the corresponding absolute coefficient is,

$$C_1 = 2[a\,(\theta_1 - \sin\theta_1) + \epsilon_0\theta_1] \tag{7·4}$$

Now when we pass from the straight line to the $s$-curve the velocity undergoes a small change $dw$. Then for the change in pressure, we have

$$dp = -\rho w\, dw.$$

But if $dw'$ denotes the change in velocity when we pass from the circle to the $\sigma$-curve, keeping on the same radius vector from the origin, we have

$$dw = dw' \quad \frac{1}{2 \sin \theta}$$

But the potential at this point of the $\sigma$-curve is exactly equal to the potential at a point of the circle, whose amplitude is $\epsilon$ less. Thus, since the old potential is

$$2V\,[\cos\theta + a \sin\theta - (a + \epsilon_0)\theta],$$

the new potential becomes,

$$2V\,[\cos(\theta - \epsilon) + a \sin(\theta - \epsilon) - (a + \epsilon_0)(\theta - \epsilon)].$$

Subtracting the old from the new and neglecting the products of small quantities, we have simply for the change in potential,

$$d\phi = 2V\epsilon \sin\theta$$

$$dw' = d/d\theta\ (d\phi) = 2V[\epsilon \cos\theta + d\epsilon/d\theta \sin\theta]$$

$$dw = V \frac{\epsilon \cos\theta + d\epsilon/d\theta \sin\theta}{\sin\theta}$$

$$dp = -\rho V^2 \frac{\epsilon \cos\theta + d\epsilon/d\theta \sin\theta}{\sin\theta}$$

But, $\epsilon = \sum B_n \cos n\theta$

and, $d\epsilon/d\theta = -\sum B_n n \sin n\theta$

so that we get for the change in pressure,

$$dp = -\rho V^2 B_n \frac{\cos n\theta \cos\theta - n \sin n\theta \sin\theta}{\sin\theta}$$

In this and the succeeding equations we are to understand summation with respect to $n$. The additional pressure resulting from the passage from the straight line to the $s$-curve is therefore,

$$P_2 = \int_{-\theta_1}^{+\theta_1} dp(-2 \sin\theta)\, d\theta$$

$$= 4q_0 B_n \int_{-\theta_1}^{+\theta_1} [\cos n\theta \cos\theta - n \sin n\theta \sin\theta]\, d\theta \qquad (7{\cdot}5)$$

$$= 4q_0 B_n I_n \qquad (7{\cdot}6)$$

where $I_n$ denotes the value of the integral in (7·5).

The corresponding absolute coefficient is

$$C_2 = P_2/q_0 t = B_n I_n. \tag{7·7}$$

We get for $I_n$ the following expression:

$$I_n = \sin(n+1)\theta_1 - \sin(n-1)\theta_1. \tag{7.8}$$

The value of $B_n$ depends of course on the shape of the $s$-curve.

In the case of the two line segments, or tail plane, which is illustrated in Fig. 4 we have,

$$s = y_1 \frac{2-x}{2} = y_1 (1 - \cos\theta), \qquad \text{for right hand line,}$$

$$s = y_2 \frac{2+x}{2} = y_2 (1 + \cos\theta), \qquad \text{for left hand line}$$

Hence we have by using (6·1)

$$\sigma = \frac{y_1}{2} \frac{1-\cos\theta}{\sin\theta} \qquad -\theta_1 \leqq \theta \leqq +\theta_1$$

$$\sigma = \frac{y_2}{2} \frac{1+\cos\theta}{\sin\theta} \qquad +\theta_1 \leqq \theta \leqq 2\pi - \theta_1.$$

By definition of $B_n$ we have

$$B_n = 1/\pi \int_0^{2\pi} \sigma \sin n\theta \, d\theta \tag{7·9}$$

Two integrations are necessary since $\sigma$ has two different analytical expressions in the interval. The integration yields:

$$B_n = 2y_1/\pi \left[-\tfrac{1}{2} \frac{\sin n\theta_1}{n} + \frac{\sin(n-1)\theta_1}{n-1} + - \cdots \mp \frac{\sin 2\theta_1}{2} \pm \sin\theta_1 \mp \tfrac{1}{2}\theta_1\right]$$

$$-2y_2/\pi \left[-\tfrac{1}{2} \frac{\sin n\theta_1}{n} + \frac{\sin(n-1)\theta_1}{n-1} + + \cdots + \frac{\sin 2\theta_1}{2} + \sin\theta_1 + \tfrac{1}{2}\theta_1 - \pi/2\right] \tag{7·10}$$

In the first brackets the upper signs will occur when $n$ is even, the lower when $n$ is odd. The two expressions for $\sigma$ coincide when $\theta = \theta_1$. Hence we have,

$$y_2 = \tan^2 \theta_1/2 \cdot y_1.$$

Thus $B_n$ will contain $y_1$ as a factor.

8. *Hinge Moment of Elevator Force.*—Another quantity of practical importance is the moment about the hinge of the forces acting on the elevator. Like the hinge force, the moment will consist of two parts $M_1$ and $M_2$, where $M_2$ denotes the additional moment experienced when we pass from the straight line to the $s$-curve. The integral expressing the value of $M_1$ is similar to that for $P_1$ except that there is the additional factor $(x - x_1)$ or $2(\cos\theta - \cos\theta_1)$ under the integral sign. Hence

$$M_1 = 4q_0 \int_{-\theta_1}^{+\theta_1} [\sin\theta - 2a\cos\theta + 2(a+\epsilon_0)][\cos\theta - \cos\theta_1]d\theta \qquad (8{\cdot}1)$$

Integrating and dividing by $q_0t^2$ to obtain the absolute coefficient, we have, since the above is for $t = 4$:

$$C_1' = a/2\ \left(\frac{\sin 2\theta_1}{2} - \theta_1\right) + (a+\epsilon_0)(\sin\theta_1 - \theta_1\cos\theta_1) \qquad (8{\cdot}2)$$

To find $M_2$ we apply the same argument as in finding $P_2$. The expression for $M_2$ will be similar to that for $P_2$, given in (7·5), except for the factor $2(\cos\theta - \cos\theta_1)$ under the integral sign. Hence we have for $M_2$:

$$M_2 = 8q_0B_n \int_{-\theta_1}^{+\theta_1} [\cos n\theta\cos\theta - n\sin n\theta\sin\theta](\cos\theta - \cos\theta_1)d\theta \cdots \qquad (8.3)$$

$$M_2 = 8q_0B_nI'_n \qquad (8{\cdot}4)$$

where $I'_n$ denotes the integral in (8·3). To get the corresponding absolute coefficient we divide by $q_0t^2$ and have, since the above is for $t = 4$,

$$C'_2 = \tfrac{1}{2}B_nI'_n \qquad (8{\cdot}5)$$

In equations (8·3), (8·4) and (8·5) there is summation with respect to $n$. The value of $I'_n$ is most easily obtained by using Euler's formulas for $\sin\theta$, $\cos\theta$, $\sin n\theta$ and $\cos n\theta$. The value thus obtained is the following:

$$I'_n = \frac{n+1}{2(n+2)}\sin(n+2)\theta_1 - \cos\theta_1\sin(n+1)\theta_1 + 1/n\ \sin\theta_1$$
$$+ \cos\theta_1\sin(n-1)\theta_1 - \frac{n-1}{2(n-2)}\sin(n-2)\theta_1. \qquad (8.6)$$

This formula holds for all values of $n$ except $n = 2$. In that case the only change occurs in the last term, which becomes:

$$-\tfrac{1}{2}(n-1)\theta_1.$$

For $n = 1$ of course the last two terms are zero.

The total absolute moment coefficient $C_m$ is now obtained as the sum of the two coefficients just deduced and we have,

$$C_m = C_1' + C'_2.$$

The values of $I_n$ and $I'_n$ do not depend on the shape of the wing. By putting $\theta_1 = \pi$ the values would apply to a wing of any shape. The $B_n$ of course depend on the shape of the wing.

## IV. NUMERICAL TABLES.

9. *Method of Computation.*—We shall now indicate briefly the procedure of computing numerical values by means of the formulas derived in the two foregoing methods. The tables which follow were computed by the method of Part II.

The Hinge Force Coefficient is, from (4·15),

$$C_l = A_1 a + C_3 + C_4.$$

From (3·1), $$A_1 = 2(\theta_1 - \sin\theta_1).$$

In the special case of the tail plane, we have from (5·5),

$$C_3 = -2\beta/\pi \ \theta_1 \sin\theta_1$$

and from (5·6), $$C_4 = 2\beta/\pi \ (\theta_1^2 + \theta_1 \sin\theta_1)$$

so that, $$C_l = 2(\theta_1 - \sin\theta_1)\, a + \ 2\theta_1^2/\pi \ \beta.$$

Taking as an illustration $a = 10° = .1745$, $\beta = 5° = .0873$ and $\theta_1 = 60° = 1.0472$, we find easily, $C_l = .1242$. This value will be found in the first table under the appropriate values of $a$ and $\beta$ and under $E/T = .25$, since in general, $E/T = \frac{1}{2}(1 - \cos\theta_1)$.

The computation using the formulas of Part III is more extended. We have, $C_l = C_1 + C_2$, where, from (7.4),

$$C_1 = 2\,[a_1(\theta_1 - \sin\theta_1) + \epsilon_0\theta_1]$$

and from (7·7), $$C_2 = \Sigma B_n I_n.$$

$B_n$ is given by (7·10), $I_n$ by (7·8) and $\epsilon_0 = \Sigma B_n$. We have written here $a_1$ in place of $a$, for the $a$ in $C_1$ has a slightly different meaning from that of Part II. The $a$ of Part II is the angle of attack with respect to the stabilizer, while the $a$ of Part III (which we now denote by $a_1$), is the angle of

attack with respect to the line joining the leading and trailing edges of the section. (See Figs. 3 and 4). We have

$$a_1 = a + \frac{1 - \cos\theta_1}{2}\beta$$

so that here, $a_1 = .1963$. Also, $y_1 = \beta = .0873$.

Following is a table of computed values,

| $n$ | $\frac{B_n}{y_1\sqrt{3}}$ | $\frac{I_n}{\sqrt{3}}$ | $\frac{B_n I_n}{y_1}$ |
|---|---|---|---|
| 1 | +.1086 | + .5 | +.1629 |
| 2 | +.0419 | — .5 | —.0629 |
| 3 | +.0025 | —1.0 | —.0075 |
| 4 | —.0111 | — .5 | +.0167 |
| 5 | —.0081 | + .5 | —.0122 |
| 6 | —.0005 | +1.0 | —.0015 |
| 7 | +.0040 | + .5 | +.0060 |
| 8 | +.0033 | — .5 | —.0050 |
| 9 | +.0002 | —1.0 | —.0006 |
| 10 | —.0020 | — .5 | +.0030 |

The alternating character of the series gives more accurarcy than is at first apparent. In fact, very little change occurs in going as far as $n = 20$. We get,

$$\Sigma \frac{B_n}{y_1\sqrt{3}} = .1388, \text{ so that } \epsilon_0 = .0210, \text{ and,}$$

$$C_1 = 2\,[.1973\,(1.0472 - .8660) + .0210 \cdot 1.0472] = .1152.$$

$$\Sigma \frac{B_n I_n}{y_1} = .0989, \text{ so that, } C_2 = .0086.$$

$$C_l = C_1 + C_2 = .1238.$$

The last decimal place is not to be relied on, since four place tables were used in the calculation. The difference in the two calculated values of $C_l$ is due to the inaccuracies which naturally creep into any extended calculation. The closeness of the two results furnishes an adequate numerical check on the two methods.

In Part II the moment is about the center of the wing, in Part III it is about the hinge. The calculations are similar to those already outlined.

*Hinge Force Coefficient.*

| $E/T$ \ $\beta$ | $\alpha = 5°$ | | | | $\alpha = 10°$ | | | |
|---|---|---|---|---|---|---|---|---|
| | 5° | 10° | 15° | 20° | 5° | 10° | 15° | 20° |
| .05 | .0140 | .0253 | .0366 | .0479 | .0166 | .0279 | .0392 | .0505 |
| .10 | .0306 | .0537 | .0766 | .0996 | .0382 | .0612 | .0842 | .1072 |
| .15 | .0494 | .0846 | .1198 | .1550 | .0636 | .0988 | .1340 | .1692 |
| .20 | .0700 | .1178 | .1656 | .2134 | .0922 | .1400 | .1878 | .2356 |
| .25 | .0925 | .1534 | .2143 | .2752 | .1242 | .1851 | .2460 | .3069 |
| .30 | .1171 | .1918 | .2665 | .3412 | .1595 | .2342 | .3089 | .3836 |
| .35 | .1437 | .2329 | .3221 | .4113 | .1981 | .2873 | .3765 | .4657 |
| .40 | .1723 | .2765 | .3807 | .4849 | .2403 | .3445 | .4487 | .5529 |
| .45 | .2033 | .3236 | .4439 | .5642 | .2863 | .4066 | .5269 | .6472 |
| .50 | .2369 | .3740 | .5111 | .6482 | .3367 | .4738 | .6109 | .7480 |
| | $\alpha = 15°$ | | | | $\alpha = 20°$ | | | |
| .05 | .0193 | .0306 | .0419 | .0532 | .0219 | .0332 | .0445 | .0558 |
| .10 | .0458 | .0688 | .0918 | .1148 | .0534 | .0764 | .0994 | .1224 |
| .15 | .0778 | .1130 | .1482 | .1834 | .0920 | .1272 | .1624 | .1976 |
| .20 | .1144 | .1622 | .2100 | .2578 | .1366 | .1844 | .2322 | .2800 |
| .25 | .1558 | 2167 | .2776 | .3385 | .1875 | .2484 | .3093 | .3702 |
| .30 | .2019 | .2766 | .3513 | .4260 | .2443 | .3190 | .3937 | .4674 |
| .35 | .2526 | .3418 | .4310 | .5202 | .3071 | .3963 | .4855 | .5747 |
| .40 | .3084 | .4126 | .5168 | .6210 | .3765 | .4817 | .5859 | .6901 |
| .45 | .3693 | .4896 | .6099 | .7302 | .4523 | .5726 | .6929 | .8132 |
| .50 | .4365 | .5736 | .7107 | .8478 | .5363 | .6734 | .8105 | .9476 |

$\alpha$ is the angle of attack estimated with reference to the stabilizer.

$\beta$ is the angle between the stabilizer and the elevator.

$E/T$ is the ratio of the length of the elevator to that of the whole tail plane.

*Coefficient of Moment about Center of Wing of Elevator Forces.*

| $E/T$ \ $\beta$ | $\alpha = 5°$ | | | | $\alpha = 10°$ | | | |
|---|---|---|---|---|---|---|---|---|
| | 5° | 10° | 15° | 20° | 5° | 10° | 15° | 20° |
| .05 | .0065 | .0118 | .0170 | .0223 | .0078 | .0131 | .0183 | .0236 |
| .10 | .0133 | .0233 | .0332 | .0432 | .0167 | .0267 | .0366 | .0466 |
| .15 | .0199 | .0340 | .0480 | .0621 | .0257 | .0398 | .0538 | .0679 |
| .20 | .0259 | .0433 | .0608 | .0783 | .0343 | .0517 | .0692 | .0867 |
| .25 | .0314 | .0517 | .0721 | .0924 | .0424 | .0627 | .0831 | .1034 |
| .30 | .0359 | .0585 | .0810 | .1036 | .0493 | .0719 | .0944 | .1170 |
| .35 | .0394 | .0634 | .0873 | .1112 | .0549 | .0789 | .1028 | .1267 |
| .40 | .0417 | .0661 | .0906 | .1151 | .0589 | .0833 | .1078 | .1323 |
| .45 | .0427 | .0670 | .0914 | .1158 | .0610 | .0853 | .1097 | .1341 |
| .50 | .0419 | .0652 | .0884 | .1117 | .0607 | .0840 | .1072 | .1305 |
| | $\alpha = 15°$ | | | | $\alpha = 20°$ | | | |
| .05 | .0090 | .0143 | .0195 | .0248 | .0101 | .0154 | .0206 | .0259 |
| .10 | .0200 | .0300 | .0399 | .0499 | .0233 | .0333 | .0432 | .0532 |
| .15 | .0315 | .0456 | .0596 | .0737 | .0373 | .0514 | .0654 | .0795 |
| .20 | .0427 | .0601 | .0776 | .0951 | .0512 | .0686 | .0861 | .1036 |
| .25 | .0534 | .0737 | .0941 | .1144 | .0644 | .0847 | .1051 | .1254 |
| .30 | .0627 | .0853 | .1078 | .1304 | .0761 | .0987 | .1212 | .1438 |
| .35 | .0704 | .0944 | .1183 | .1422 | .0860 | .1100 | .1339 | .1578 |
| .40 | .0761 | .1005 | .1250 | .1495 | .0933 | .1177 | .1422 | .1667 |
| .45 | .0794 | .1037 | .1281 | .1525 | .0977 | .1220 | .1464 | .1708 |
| .50 | .0794 | .1027 | .1259 | .1492 | .0982 | .1215 | .1447 | .1680 |

$\alpha$, $\beta$, and $E/T$ have the same meaning as in the previous table.

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# The Correspondence Between the Tangent Plane of a Surface and its Point of Contact.

By Ernest P. Lane.

## 1. Introduction.

The correspondence between the tangent plane of a quadric surface and its point of contact is a symmetric polar correlation of space. In the literature of projective differential geometry there are instances of the intuitive use of the more general correspondence between the tangent plane of an analytic surface and its point of contact. It is the purpose of this paper to furnish an analytic formulation of this correspondence and then to make some applications thereof. In particular, this correspondence is used to define a certain quartic birational transformation between points in the tangent plane of a surface and planes through the contact point.

## 2. Analytic Formulation.

Let the homogeneous coordinates $y^{(1)}, \cdots, y^{(4)}$ of a general point $P_y$ on a non-degenerate non-developable surface $S_y$ be given as analytic functions of two independent variables $u, v$. If $S_y$ is referred to its asymptotic net, the four functions $y$ are solutions of a completely integrable system of differential equations which can be reduced * to the form

$$y_{uu} + 2by_v + fy = 0,$$
$$y_{vv} + 2a'y_u + gy = 0.$$

The four points $y, y_u, y_v, y_{uv}$ may be used as the vertices of a local tetrahedron of reference, which we shall denote by $T_p$. The usual agreement, that a point represented in the original coordinate system by an expression of the form

$$x_1y + x_2y_u + x_3y_v + x_4y_{uv}$$

shall have its local coordinates proportional to $x_1, \cdots, x_4$, implies that the unit point for the local system is the point represented by $y + y_u + y_v + y_{uv}$ in the original system. Then the local coordinates of a point $y_1$ on $S_y$ near $P_y$ are given† by the following expansions:

* Wilczynski, "First Memoir," *Transactions of the American Mathematical Society*, Vol. 8 (1907), p. 246.

† Wilczynski, "Second Memoir," *Transactions of the American Mathematical Society*, Vol. 9 (1908), p. 100.

$$
\begin{aligned}
x_1 &= 1 - \tfrac{1}{2} f\,\Delta u^2 - \tfrac{1}{2} g\,\Delta v^2 + \cdots, \\
x_2 &= \Delta u - a'\Delta v^2 + \cdots, \\
x_3 &= \Delta v - b\,\Delta u^2 + \cdots, \\
x_4 &= \Delta u \Delta v + \cdots,
\end{aligned} \tag{1}
$$

The coordinates of the tangent plane $\Pi_y$ of $S_y$ at $P_y$ are proportional to the four third order determinants in the matrix

$$(y^{(i)},\ y_u^{(i)},\ y_v^{(i)}), \quad (i = 1, 2, 3, 4).$$

If $\eta$ is a general one of these, we may write

$$\eta = (y,\ y_u,\ y_v). \tag{2}$$

Then by differentiation and elimination we obtain

$$
\begin{aligned}
&\eta_u = (y,\ y_u,\ y_{uv}), \quad \eta_v = (y,\ y_{uv},\ y_v), \\
&\eta_{uv} = (y_u,\ y_{uv},\ y_v) + 4a'b\eta,
\end{aligned}
$$

and find that $\eta$ is a solution * of the system of equations

$$
\begin{aligned}
&\eta_{uu} + 2B\eta_v + F\eta = 0, \\
&\eta_{vv} + 2A'\eta_u + G\eta = 0,
\end{aligned} \tag{4}
$$

in which

$$A' = -a', \quad B = -b, \quad F = f + 2b_v, \quad G = g + 2a'_u. \tag{5}$$

Let us consider the tetrahedron $T_\pi$, whose faces are the planes $\eta$, $\eta_u$, $\eta_v$, $\eta_{uv}$. Inspection of equations (2) and (3) shows that the vertices of this tetrahedron are the points $y$, $y_u$, $y_v$, $y_{uv} - 4a'by$. Therefore, if $\bar{x}_1, \cdots, \bar{x}_4$ are the coordinates of a point referred to $T_\pi$ and if $x_1, \cdots, x_4$ are the coordinates of the same point referred to $T_p$, then the equations for transformation of point coordinates between $T_\pi$ and $T_p$ are

$$\bar{x}_1 = x_1 + 4a'bx_4, \quad \bar{x}_2 = x_2, \quad \bar{x}_3 = x_3, \quad \bar{x}_4 = x_4.$$

The equations for transformation of plane coordinates between $T_\pi$ and $T_p$ are

$$\bar{u}_1 = u_1, \quad \bar{u}_2 = u_2, \quad \bar{u}_3 = u_3, \quad \bar{u}_4 = -4a'bu_1 + u_4. \tag{6}$$

Since the unit plane of $T_\pi$ passes through the three points whose coordinates, referred to $T_\pi$, are

$$(1, -1, 0, 0), \quad (0, 1, -1, 0), \quad (0, 0, 1, -1),$$

---

* Wilczynski, "First Memoir," p. 259. Here it is incorrectly stated that $F = f$, $G = g$.

it follows that this plane may be regarded as determined by the three points

$$y - y_u, \quad y_u - y_v, \quad y_v - y_{uv} + 4a'by.$$

Then a short calculation shows that the unit plane of $T_\pi$ is the plane represented in the original coordinate system by the expression $\eta - \eta_u - \eta_v + \eta_{uv}$. Moreover, the faces $\eta$, $\eta_u$, $\eta_v$, $\eta_{uv}$ of $T_\pi$ are respectively the planes whose equations, referred to $T_\pi$, are

$$\bar{x}_4 = 0, \quad \bar{x}_3 = 0, \quad \bar{x}_2 = 0, \quad \bar{x}_1 = 0.$$

Therefore *an expression of the form*

$$\bar{u}_4\eta + \bar{u}_3(-\eta_u) + \bar{u}_2(-\eta_v) + \bar{u}_1\eta_{uv}$$

*represents a plane whose local coordinates referred to* $T_\pi$ *are* $\bar{u}_1$, $\bar{u}_2$, $\bar{u}_3$, $\bar{u}_4$.

Let us consider the tangent plane $\eta_1$ of $S_y$ at the point $y_1$, and seek to obtain power series expansions for the local coordinates of this plane. Denoting the coordinates of $\eta_1$, referred to $T_\pi$, by $\bar{u}_1, \cdots, \bar{u}_4$, we observe that the expansions for $\bar{u}_4, -\bar{u}_3, -\bar{u}_2, \bar{u}_1$, expressed in terms of $A'$, $B$, $F$, $G$, must have the same form respectively as the series for the coordinates $x_1, \cdots, x_4$ of $y_1$, given in equations (1), expressed in terms of $a'$, $b$, $f$, $g$, and referred to $T_p$. Then from equations (5) and (6) we obtain the following expansions for the coordinates $u_1, \cdots, u_4$ of the tangent plane $\eta_1$, expressed in terms of $a'$, $b$, $f$, $g$, and referred to $T_p$:

$$\begin{aligned}
u_1 &= \Delta u\,\Delta v + \cdots, \\
u_2 &= -\Delta v - b\Delta u^2 + \cdots, \\
u_3 &= -\Delta u - a'\Delta v^2 + \cdots, \\
u_4 &= 1 - \tfrac{1}{2}[(f + 2b_v)\Delta u^2 - 8a'b\Delta u\Delta v + (g + 2a'_u)\Delta v^2] + \cdots.
\end{aligned}$$

It appears from the foregoing discussion that *the correspondenec between the coordinates* $x_1, \cdots, x_4$ *of a point* $y_1$ *on* $S_y$ *near* $P_y$ *and the coordinates* $u_1, \cdots, u_4$ *of the tangent plane* $\eta_1$ *of* $S_y$ *at the point* $y_1$, *with reference to the tetrahedron* $T_p$ *whose vertices are the points* $y$, $y_u$, $y_v$, $y_{uv}$ *and whose unit point is the point* $y + y_u + y_v + y_{uv}$, *is represented by the equations:*

$$(7) \qquad x_1 = -4a'bu_1 + u_4, \quad x_2 = -u_3, \quad x_3 = -u_2, \quad x_4 = u_1,$$

*accompanied by the substitution*

$$(S) \qquad \begin{pmatrix} a', & b, & f, & g \\ -a', & -b, & f + 2b_v, & g + 2a'_u \end{pmatrix}.$$

The transformation (7) accompanied by the substitution $(S)$ will be referred to as *duality*. It is easy to verify that this dualistic transformation has the following properties.

1. To a point in the tangent plane corresponds a plane through the contact point.

2. To a point on an asymptotic tangent corresponds a plane through the same tangent.

3. To a point on a tangent invariant under $(S)$ corresponds a plane through the conjugate tangent.

## 3. The Osculating Quadric.

The equation of the osculating quadric,* or quadric of Lie, of $S_y$ at $P_y$, referred to $T_p$, is

$$x_1x_4 - x_2x_3 + 2a'bx_4^2 = 0. \tag{8}$$

*The transformation* (7) *unaccompanied by the substitution* $(S)$ *is precisely the correspondence between a point and its polar plane with respect to the osculating quadric. Furthermore, if a configuration is invariant under* $(S)$, *the dual configuration is obtained by polar reciprocation with respect to the osculating quadric.*

Every quadric whose equation can be written in the form

$$x_1x_4 - x_2x_3 + 2\kappa a'bx_4^2 = 0, \tag{9}$$

where $\kappa$ is a constant, is tangent to the osculating quadric at every point of the two generators through $P_y$, and is invariant under $(S)$. The equation of the dual quadric is easily found to be

$$x_1x_4 - x_2x_3 + 2(2-\kappa)a'bx_4^2 = 0. \tag{10}$$

The quadrics (9) and (10) are reciprocal quadrics with respect to the osculating quadric, and reduce to the osculating quadric if $\kappa = 1$. Let us consider a line $l'$ through $P_y$ but not in $\Pi_y$. Such a line joins the point $y(1, 0, 0, 0)$ and a point $z(0, -a, -\beta, 1)$. This line meets at $P_y$ all of the quadrics under consideration. It meets the osculating quadric also at the point

$$P_1(a\beta - 2a'b, -a, -\beta, 1);$$

* Wilczynski, "Second Memoir," p. 82.

it meets the quadric (9) at the point

$$P_\kappa(a\beta - 2\kappa a'b, -a, -\beta, 1),$$

and meets the quadric (10) at the point

$$P_{2-\kappa}(a\beta - 2(2-\kappa)a'b, -a, -\beta, 1).$$

It follows that the points $P_\kappa$ and $P_{2-\kappa}$ separate harmonically the points $P_y$ and $P_1$ so that, for every value of $\kappa$ and for every line $l'$,

$$(P_\kappa, P_{2-\kappa}, P_y, P_1) = -1.$$

The equation of the canonical * quadric of $S_y$ at $P_y$ is

$$x_1x_4 - x_2x_3 = 0.$$

The line $l'$ meets this quadric at $P_y$ and also as the point

$$P_0(a\beta, -a, -\beta, 1).$$

Calculation of the anharmonic ratio of the points $P_y$, $P_0$, $P_1$, $P_\kappa$ gives

$$(P_y, P_0, P_1, P_\kappa) = \kappa. \tag{11}$$

Therefore *any particular quadric of the pencil* (9) *is distinguished geometrically from the other quadrics of the pencil by the fact that, for every line* $l'$, *equation* (11) *is valid.*

## 4. ČECH'S TRANSFORMATION.

Let us consider the pencil of cubic curves in $\Pi_y$ whose equations referred to $T_p$ are

$$\tfrac{2}{3}\kappa(bx_2^3 + a'x_3^3) - x_2x_3(x_1 + \beta x_2 + ax_3) = x_4 = 0. \tag{12}$$

Every cubic of this pencil has all of the following properties.

1. There is a double point at $P_y$ (1, 0, 0, 0).

2. The double-point tangents are the asymptotic tangents, whose equations are respectively $x_3 = x_4 = 0$ and $x_2 = x_4 = 0$.

3. The three inflexion points are on the tangents of Darboux, whose equations are $bx_2^3 + a'x_3^3 = x_4 = 0$.

4. The three inflexions are on the line whose equations are $x_1 + \beta x_2 + ax_3 = x_4 = 0$.

---

* Wilczynski, "Second Memoir," p. 112.

These properties are characteristic of the pencil.

Čech has considered * a one-parameter family of cubic birational transformations,

$$(\Sigma_\kappa) \qquad \begin{aligned} u_1 &= 0, \\ \sigma\, u_2 &= x_2 x_3^2, \\ \sigma\, u_3 &= x_2^2 x_3, \\ \sigma\, u_4 &= \tfrac{2}{3}\kappa(b x_2^3 + a' x_3^3) - x_1 x_2 x_3. \end{aligned}$$

*The inverse of a transformation $\Sigma_\kappa$ is its dual.* In the memoir cited Čech has employed the metric notion of curvature. Since we shall need to make use of certain particular transformations $\Sigma_\kappa$, we may be permitted to set forth briefly a purely projective description of the class of all transformations $\Sigma_\kappa$ and to show how to distinguish projectively any particular transformation from all others.

It is not difficult to show that every transformation $\Sigma_\kappa$ has all of the following properties.

1. It is a one-to-one correspondence between points in the tangent plane and planes through the contact point.

2. To every plane through the contact point, distinct from the tangent plane, corresponds a point on the line conjugate to the line of intersection of the given plane and the tangent plane.

3. For every line $l'$ through the contact point but not in the tangent plane, it is true that the planes of the pencil with $l'$ as axis correspond to points on a cubic of the pencil (12), the line of inflexions being the reciprocal polar of $l'$ with respect to the osculating quadric (8).

We shall now show that every transformation having these three properties is a transformation $\Sigma_\kappa$. First of all, for a transformation having the first property, we see that $u_1 = x_4 = 0$. In the second place, any plane through $P_y$ intersects $\Pi_y$ in the line whose equations are $u_2 x_2 + u_3 x_3 = x_4 = 0$. The equations of the conjugate line are $u_2 x_2 - u_3 x_3 = x_4 = 0$. Therefore a transformation having the first two properties must be of the form

$$\begin{aligned} u_1 &= 0, \\ \sigma\, u_2 &= x_3, \\ \sigma\, u_3 &= x_2, \\ \sigma\, u_4 &= f(x_1, x_2, x_3), \end{aligned}$$

* Čech, "L'intorno di un punto di una superficie considerato dal punto di vista proiettivo," *Annali di Matematica*, Vol. 31 (1922), p. 192.

where $f$ is a function that can be determined by the third property. Any line $l'$ through $P_y$ but not in $\Pi_y$ may be regarded as determined by a point $z(0, -a, -\beta, 1)$. Any plane through $l'$ has coordinates satisfying the equations

$$u_1 = 0, \quad u_4 = au_2 + \beta u_3.$$

Therefore the corresponding point must have coordinates satisfying the equations

$$x_4 = 0, \quad f(x_1, x_2, x_3) = \beta x_2 + a x_3.$$

This point must have coordinates also satisfying the equation

$$\tfrac{2}{3}\kappa(bx_2^3 + a'x_3^3) - x_2x_3(x_1 + \beta x_2 + ax_3) = 0.$$

Eliminating $a$ and $\beta$, we determine the function $f$, so that

$$f = \frac{1}{x_2x_3}\,[\tfrac{2}{3}\kappa(bx_2^3 + a'x_3^3) - x_1x_2x_3].$$

Therefore every transformation having the three properties stated is a transformation $\Sigma_\kappa$.

We shall next show how to distinguish projectively any particular transformation $\Sigma_\kappa$ from all others. The transformation $\Sigma_0$ is the correspondence between any point in the tangent plane and its polar plane with respect to the osculating * quadric. The transformation $\Sigma_1$ is the correspondence † between any point in the tangent plane and its polar plane with respect to the quadric of Moutard associated with the tangent through the given point. The point $P_\kappa$ corresponding to any given plane $(0, u_2, u_3, u_4)$ in any particular $\Sigma_\kappa$ has for its coordinates

$$[\tfrac{2}{3}\kappa(a'u_2^3 + bu_3^3) - u_2u_3u_4,\ u_2u_3^2,\ u_2^2u_3,\ 0].$$

The points $P_0$ and $P_1$, corresponding to the said plane in $\Sigma_0$ and $\Sigma_1$ respectively, have coordinates easily written. Then the cross-ratio of the four points $P_y$, $P_0$, $P_1$, $P_\kappa$ is $\kappa$, since

$$(\infty, 0, 1, \kappa) = \kappa.$$

Therefore, *to construct the point corresponding to a given plane in a given $\Sigma_\kappa$, we may proceed as follows. First draw the line conjugate to the line of intersection of the given plane with the tangent plane. On this conjugate line mark the points $P_0$ and $P_1$ which are respectively the poles of the given plane*

* Čech, *loc. cit.*, p. 192.

† Čech, *loc. cit.*, p. 193.

*with respect to the osculating quadric and with respect to the quadric of Moutard belonging to this line. Then the point $P_\kappa$ corresponding to the given plane in the given $\Sigma_\kappa$ is the point on this line such that*

$$(P_y, P_0, P_1, P_\kappa) = \kappa.$$

## 5. A Quartic Birational Transformation.

The *directrix* of the first kind crosses one asymptotic tangent at the point $D_1(-a', 0, 1, 0)$ and crosses the other at the point $D_2\ (-\beta', 1, 0, 0)$, the functions $a', \beta'$ being defined by the formulas

$$a' = \tfrac{1}{2}\ b_v/b,\ \beta' = \tfrac{1}{2}\ a'_u/a'.$$

Similarly, the *canonical edge* of the first kind crosses the asymptotic tangents at the points $E_1(-a'', 0, 1, 0)$ and $E_2(-\beta'', 1, 0, 0)$, the functions $a'', \beta''$ being defined by

$$a'' = -\tfrac{1}{4}\ a'_v/a',\ \beta'' = -\tfrac{1}{4}\ b_u/b.$$

There is a pencil of quartic curves in the tangent plane at a point $P_y$ of a surface $S_y$, each of which has all of the following properties.

1. There is a triple point at $P_y\ (1, 0, 0, 0)$.
2. The triple-point tangents are the tangents of Darboux,

$$bx_2^3 + a'x_3^3 = x_4 = 0.$$

3. The curve passes through $E_1$ tangent to the line $E_1D_2$, whose equations are

$$x_1 + \beta'x_2 + a''x_3 = x_4 = 0.$$

4. The curve passes through $E_2$ tangent to the line $E_2D_1$, whose equations are

$$x_1 + \beta''x_2 + a'x_3 = x_4 = 0.$$

The equations of a general quartic of this pencil may be written in the form

$$\kappa\, a'bx_2^2x_3^2 + bx_2^3(x_1 + \beta''x_2 + a'x_3) + a'x_3^3(x_1 + \beta'x_2 + a''x_3) = x_4 = 0. \tag{13}$$

This quartic is of order four and class six, with one triple-point, no cusps, four double-tangents, and six inflexions.

The Hessian of the quartic (13) is a sextic

$$(14)\qquad \kappa(b^2x_2^6 - 4a'bx_2^3x_3^3 + a'^2x_3^6) + 3x_2x_3\{bx_2^3[x_1 + \beta'x_2 + (2a'' - a')x_3] + a'x_3^3[x_1 + (2\beta'' - \beta')x_2 + a'x_3]\} = 0.$$

Eliminating $x_1$ between equations (13) and (14), we find that the directions from $P_y$ to the six inflections are given by

$$(15)\qquad \kappa(b_2x_2^6 - 7a'bx_2^3x_3^3 + a'^2x_3^6) + 3x_2x_3\{bx_2^3[(\beta' - 3\beta'')x_2 + 2(a'' - 2a')x_3] + a'x_3^3[2(\beta'' - 2\beta')x_2 + (a' - 3a'')x_3]\} = 0.$$

There is an inflexion on the line $x_2 = x_4 = 0$ if, and only if, $\kappa = 0$. Then there is also an inflexion on the line $x_2 = x_4 = 0$. Therefore *the quartic* (13) *for which* $\kappa = 0$ *is the only quartic of the pencil that has inflexions at* $E_1$ *and* $E_2$.

We shall now show how to characterize that quartic (13) for which $\kappa = 1$. Consider, with Čech, any line $r$ meeting $\Pi_y$ at a point $P$ and determining with the line $P_yP$ a plane $\Pi$. The reciprocal polar of $r$ with respect to the Moutard quadric for $P_yP$ may be denoted by $r'$; the reciprooal polar of $r$ with respect to the Moutard quadric for the tangent conjugate to $P_yP$ by $r''$; and the recipprocal polar of $r$ with respect to the osculating quadric, by $r_0$. The three lines $r'$, $r''$, $r_0$ determine a quadric surface. One of the two generators of this quadric that pass through $P_y$ is the tangent conjugate to $P_yP$. The other is a line which determines with $P_yP$ a plane $\overline{\Pi}$. The coincident planes in the correspondence between $\Pi$ and $\overline{\Pi}$ envelope a cone whose equations * are

$$u_1 = 10a'^2u_2^6 + 8a'bu_2^3u_3^3 + 10bu^2u_3^6 + 12u_2u_3u_4(a'u_2^3 + bu_3^3) + 3u_2u_3(a'_vu_2^4 - 2a'_uu_2^3u_3 - 2b_vu_2u_3^3 + b_uu_3^4) = 0.$$

*The transform of this cone by means of* $\Sigma_{-\frac{5}{4}}$ *is the quartic* (13) *for which* $\kappa = 1$, together with the asymptotic tangent.

*Having characterized the quartics* $q_0$, $q_1$ *for which* $\kappa = 0$ *and* $\kappa = 1$, *we may characterize the quartic* $q_\kappa$ *for any particular value of* $\kappa$ *by observing that any line in the tangent plane through* $P_y$ *cuts* $q_0$, $q_1$, $q_\kappa$ *in three points* $P_0$, $P_1$, $P_\kappa$ *such that*

$$(P_y,\ P_0,\ P_1,\ P_\kappa) = \kappa.$$

We are now in position to define a certain one-parameter family of quartic Cremona transformations between points in the tangent plane of a

* Čech, *loc. cit.*, p. 196.

surface and planes through the contact point. Any line through $P_y$ in the tangent plane may have its equations written in the form

$$x_4 = x_3 - hx_2 = 0.$$

This line intersects the quartic $q_\kappa$, represented by equations (13), in a point $P_\kappa$ whose coordinates are given by the equations

$$\begin{aligned} \sigma\, x_1 &= -\kappa a'bh^2 - a''a'h^4 - \beta'a'h^3 - a'bh - \beta''b, \\ \sigma\, x_2 &= a'h^3 + b, \\ \sigma\, x_3 &= h(a'h^3 + b), \\ x_4 &= 0. \end{aligned} \tag{16}$$

To the quartic $q_\kappa$ corresponds by a duality a cone $C_\kappa$, whose equations are

$$\begin{aligned} u_1 = \kappa a'bu_2^2u_3^2 &+ bu_3^3(u_4 - \beta''u_3 - a'u_2) \\ &+ a'u_2^3(u_4 - \beta'u_3 - a''u_2) = 0. \end{aligned} \tag{17}$$

The coordinates of that tangent plane $\Pi_\kappa$ of the cone $C_\kappa$ which passes through the conjugate of the line $P_yP_\kappa$ are given by the equations

$$\begin{aligned} u_1 &= 0, \\ \sigma\, u_2 &= h(a'h^3 + b), \\ \sigma\, u_3 &= a'h^3 + b, \\ \sigma\, u_4 &= -\kappa a'bh^2 + a''a'h^4 + \beta'a'h^3 + a'bh + \beta''b. \end{aligned} \tag{18}$$

The correspondence between $P_\kappa$ and $\Pi_\kappa$ will be denoted by $T_\kappa$ and its equations are obtained by eliminating $h$ between equations (16) and (18):

$$\begin{aligned} \rho x_1 &= -2\kappa a'bu_2^2u_3^2 - u_4(a'u_2^3 + bu_3^3), & u_1 &= 0, \\ \rho x_2 &= u_3(a'u_2^3 + bu_3^3), & \sigma u_2 &= x_3(bx_2^3 + a'x_3^3), \\ \rho x_3 &= u_2(a'u_2^3 + bu_3^3), & \sigma u_3 &= x_2(bx_2^3 + a'x_3^3), \\ x_4 &= 0, & \sigma u_4 &= -2\kappa a'bx_2^2x_3^2 - x_1(bx_2^3 + a'x_3^3), \end{aligned} \tag{$T_\kappa$}$$

*In order to construct the plane $\Pi_u$ corresponding by $T_\kappa$ to any given point $P_x$ in the tangent plane we may proceed as follows. First draw the quartic $q_\kappa$ that passes through $P_\kappa$. Then construct the cone $C_\kappa$ dual to $q_\kappa$. Draw the tangent plane of $C_\kappa$ that passes through the conjugate of the line $P_yP_x$. This is the required plane.*

That transformation $T_\kappa$ for which $\kappa = 0$, is the pole-polar correspondence with respect to the osculating quadric. *The inverse of a transformation $T_\kappa$ is its dual.*

Let us investigate the locus of points corresponding by $T_\kappa$ to the planes

of a pencil. Consider a line $l'$ through $P_y$ but not in $\Pi_y$. The coordinates of a plane through this line satisfy the equations

$$u_1 = 0, \quad u_4 = \alpha u_2 + \beta u_3.$$

*Therefore the locus of points corresponding to the planes of the pencil with $l'$ as axis is a quartic curve whose equations are*

$$(19) \qquad 2\kappa a' b x_2^2 x_3^2 + (b x_2^3 + a' x_3^3)(x_1 + \beta x_2 + \alpha x_3) = x_4 = 0.$$

For every value of $\kappa$, this quartic has all of the following properties.

1. There is a triple point at $P_y$.
2. The triple-point tangents are the Darboux tangents.
3. The curve is tangent to the reciprocal polar of $l'$ with respect to the osculating quadric, namely, the line

$$x_1 + \beta x_2 + \alpha x_3 = x_4 = 0,$$

at the points where this line crosses the asymptotic tangents.

In particular, if the line $l'$ is the line of intersection of the osculating planes of the Segre curves at $P_y$, so that

$$\alpha = -\tfrac{1}{6}\left(\frac{a'_v}{a'} - \frac{b_v}{b}\right), \qquad \beta = -\tfrac{1}{6}\left(\frac{b_u}{b} - \frac{a'_u}{a'}\right)$$

and if $\kappa = 2$, then the quartic (19) is *a covariant quartic* which can also be defined as follows. There is * in the tangent plane a unique conic $c_1 = 0$ which passes through $P_y$ tangent to the asymptotic tangent $x_3 = x_4 = 0$ and also passes through the three ray-points of $P_y$ with respect to the Darboux curves through $P_y$. Similarly, there is another unique conic $c_2 = 0$ which passes through $P_y$ tangent to the other asymptotic tangent and passes through the same three ray-points. *The only quartic of the pencil*

$$a' x_3^2 c_1 + \kappa b x_2^2 c_2 = 0$$

*which has at $P_y$ the Darboux tangents for triple-point tangents is precisely the quartic in question.*

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* Čech, *loc. cit.*, p. 204.

# On Multiple Iterated Integrals.

By H. J. Ettlinger.

The necessary and sufficient condition that the multiple $R$-integral of a bounded function, $\int_{S^{(m)}} f(P^{(m)})dP^{(m)}$, exist, where $S^{(m)}$ is a hypercube in $m$ dimensions, is that the function be continuous "almost everywhere" (except for a null set) in $S^{(m)}$. It has also been shown that the existence of the multiple integral carries with it the existence of the associated iterated integrals and all of these are equal in value to the multiple integral.*

W. H. Young † and Lichtenstein ‡ have given sufficient conditions for the existence of the iterated integrals for the case $m = 2$, without implying the existence of the double integral. Lichtenstein ‡ and Gillespie § have given sufficient conditions for the equality of the two two-fold iterated integrals. The writer ¶ has extended these results and has given a necessary and sufficient condition for the existence and equality of the two two-fold iterated integrals.

It is our present purpose to extend the results obtained for the case of functions of two variables to functions of $m$ variables, and to obtain other properties of $m$-fold iterated $R$-integrals. The results of this paper are used to establish the existence of the unique integrable solution of the Fredholm integral equation.‖

## Preliminary Theorems.

Throughout this paper, the functions are real functions of real variables. The integral variable for the infinite range 1, 2, 3, · · · is represented by $n$.

* For a summary of the results and a bibliography of the original sources, see Hobson, *The Theory of Functions of a Real Variable*, second edition, Macmillan, London, 1921, pp. 446-452, pp. 573-578.

See also *Encyclopädie der Mathematischen Wissenschaften*, Band II 3, Heft 7, pp. 1115-1119.

† "On parametric integration," *Monatshefte für Mathematik und Physik*, Vol. 2 (1910), p. 127.

‡ "Über die Integration eines Bestimmten Integrals in Bezug auf einen Parameter," *Goettinger Nachrichten*, 1910, pp. 468-470.

§ "Repeated Integrals," *Annals of Mathematics*, (2) Vol. 20 (1919), pp. 224-225.

¶ "On the inversion of the order of integration of a two-fold iterated integral," accepted for publication by the *Bulletin of the American Mathematical Society*.

‖ "On the Fredholm integral equation," offered to the *Tohoku Mathematical Journal*.

Let $I = (a, b)$ be divided into $n$ subintervals $I_{in} = (t_{i-1,n}, t_{in})$ of length $\Delta_{in}$ by means of the $n+1$ distinct points $a \equiv t_{0n}, t_{1n}, \cdots, t_{nn} \equiv b$. Let $\max \Delta(n)$ represent the greatest $\Delta_{in}$ for each $n$. Then the sets of subdivision $I_{in}$, represent a *fine* or *complete* subdivision of $I$, if $\lim_{n\to\infty} \max \Delta(n) = 0$. Let $\bar{x}$ be any fixed $x$ in $I$, then $I_n(\bar{x})$ will designate a subinterval $I_{in}$ which for each value of $n$ contains $\bar{x}$. If $\Delta_n(\bar{x})$ is the length of $I_n(\bar{x})$, then for a *fine* subdivision, $\lim_{n\to\infty} \Delta_n(\bar{x}) = 0$ for each $x$ in $I$.

For a given set of subintervals $I_{in}$ of $I$, $h_n(x)$ is a horizontal function of index $n$ on $I$, if $h_n(x)$ has a constant value $h_{in}$ for each value of $x$ interior to $I_{in}$. Let the sum $A_n = \sum_{i=1}^{n} h_{in}\Delta_{in}$ be the *area* under $h_n(x)$ on $I$.

The following theorem is equivalent to the Duhamel-Moore * theorem in one dimension.

THEOREM I.

*Hypothesis.* 1. $h_n(x)$ is an infinite sequence of horizontal functions on index $n$ on $I$.

2. $|h_n(x)| < K$ for all values of $n$ and all values of $x$ on $I$, where $K$ is independent of $n$ and $x$.

3. If $M_0$ is a null set on $I$, and $x$ is a value on $I$ not in $M_0$, $\lim_{n\to\infty} h_n(x) = 0$.

4. $A_n$ is the area on $I$ under $h_n(x)$.

*Conclusion.* $\lim_{n\to\infty} A_n = 0$.

Proof. Given $\epsilon > 0$, let $I_{n\epsilon}$ be the totality of those subintervals $I_{in}$, for which $|h_n(x)| > \epsilon$. Let the length of these subintervals be $L(I_{n\epsilon})$. Given $\delta > 0$, there exists $N_{\delta\epsilon} > 0$, such that $L(I_{n\epsilon}) < \delta$ for $n > N_{\delta\epsilon}$. Suppose the latter statement false then $L(I_{n\epsilon}) > \delta > 0$ for an infinite set of values of $n$. By a theorem due to W. H. Young † it follows that there exists a set of points $\bar{I}$ of measure not less than $\delta$, each of whose points is common to an infinite number of intervals $I_{n\epsilon}$. Let $\bar{x}$ be such a point not in $M_0$, then $\lim_{n\to\infty} h_n(\bar{x})$

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* R. L. Moore, "On Duhamel's Theorem," *Annals of Mathematics* (2), Vol. 16 (1911), pp. 45-49. This paper will be referred to as Moore, D. T.

† "Open Sets and the Theory of Content," *Proceedings of the London Mathematical Society* (2), Vol. 2 (1905), p. 25.

cannot be zero. This contradicts hypothesis 3. Hence the second statement is true and

$$| A_n | < \epsilon(b - a) + K\delta$$

holds for all values of $n > N_{\epsilon\delta}$. Hence $\lim_{n\to\infty} A_n = 0$.

THEOREM II. FUNDAMENTAL THEOREM OF SUMMATION.

*Hypothesis.* 1. $f(x)$ is bounded and integrable on $I$.

2. $H_n(x)$ is an infinite sequence of horizontal functions of index $n$ on $I$, associated with a *fine* subdivision of $I$.

3. $H_n(x)$ is bounded for all values of $n$ and all values of $x$ in $I$.

4. $A_n$ is the area on $I$ under $H_n(x)$.

5. $\lim_{n\to\infty} H_n(x) = f(x)$ for every fixed $x$ on $I$ except for a null set.

*Conclusion.* $\lim_{n\to\infty} A_n = \int_a^b f(x)\, dx$.

Proof. Identify $h_n(x)$ of Theorem I as $H_n(x) - f_n(x)$, where $f_n(x)$ is a horizontal function of index $n$ on $I$, such that $f_n(x) = f(t_{in})$ for all values of $x$ interior to $I_{in}$ and $t_{in}$ is any value of $x$ in $\Delta_{in}$. From hypotheses 1 and 3 it follows that $H_n(x) - f_n(x)$ is bounded. By hypothesis 1 $f(x)$ is also integrable and hence continuous "almost everywhere." Hence from hypotheses 2 and 5, $H_n(x) - f_n(x)$ converges to zero "almost everywhere" as $n$ increases without limit. Hence by Theorem I we have the present conclusion.

Let $P^{(m)} = (x^{(1)}, x^{(2)}, \cdots, x^{(m)})$ be a set of values in an $m$-dimensional cell (hypercube) $S^{(m)}$ defined by the inequalities, $I = (a < x^{(i)} < b)$, $i = 1, 2, \cdots, m$. Let $C_0^{(i)}$ be a discrete * set of values of $x^{(i)}$ on $I$, and let $C_0$ represent the totality of points $P^{(m)}$ at least one of whose coordinates lies in a $C_0^{(i)}$. Then $C_0$ can be regarded as the sum of $m$ discrete sets on $I$, and is therefore itself discrete, or it may be regarded as a discrete set in $S^{(m)}$. A point $P^{(m-j)}$ does not lie in $C_0$, if no one of its coordinates $x^{(i)}$ lies in $C_0^{(i)}$. Let a section of $S^{(m)}$ by the hyperplanes $x = x^{(i_1)}, \cdots, x = x^{(i_j)}$ be $S^{(m-j)i_1 \cdots i_j}$ and let $P^{(m-j)i_1 \cdots i_j}$ be a point therein. Let $dP^{(m)} = dx^{(1)}dx^{(2)} \cdots dx^{(m)}$.

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* A *discrete* point set is a set of content zero, *i. e.* it can be enclosed in a *finite* number of intervals, the sum of whose lengths is as small as you please. This term is due to Harnack. See Pierpont, *The Theory of Functions of Real Variables*, Ginn and Company, New York, 1905, Vol. 1, p. 355. This kind of set is called by some writers an *integrable* set after du Bois-Reymond.

## FUNDAMENTAL THEOREM ON ITERATION.

THEOREM III.

*Hypothesis.* 1. $f(P^{(m)})$ is a bounded function of $P^{(m)}$ in $S^{(m)}$, i. e., $|f(P^{(m)})| < K$ for all points $P^{(m)}$ in $S^{(m)}$, where $K$ is a constant independent of $P^{(m)}$.

2. $f(P^{(m)})$ is $R$-integrable in $x^{(i)}$ on $I$, for every fixed $P^{(m-1)i}$ not on $C_0$.

*Conclusion.* 1. $F^{(j)}(x^{(k)}, \overline{P}^{(m-j-1)}) = \int_a^b \cdots \int_a^b f(x^{(k)}, \overline{P}^{(m-j-1)}, P^{(j)})\, dP^{(j)}$

is an integrable function of $x^{(k)}$ on $I$ for every fixed $\overline{P}^{(m-j-1)}$ not in $C_0$.

2. All the $m$-fold iterated integrals $\int_a^b dt^{(i_1)} \int_a^b dt^{(i_2)} \cdots f(P^{(m)})\, dt^{(i_m)}$ exist.

3. All the $m!$ $m$-fold iterated integrals are equal in value.

Proof. To establish conclusion 1 consider

$$(1) \qquad F^{(1)}(x^{(i_2)}, \overline{P}^{(m-2)}) = \int_a^b f(x^{(i_1)}, x^{(i_2)}, \overline{P}^{(m-2)})\, dx^{(i_1)}$$

where $\overline{P}^{(m-2)} = \overline{P}^{(m-2)i_1 i_2}$ is not in $C_0$.

Let $I_{in}$ define a *fine* subdivision of $I$ and $t_{in}$ a value of $x^{(i_2)}$ on $I_{in}$, where $(x^{(i_1)}, t_{in})$ is not in $C_0$. Then

$$(2) \qquad \sum_{i=1}^{n} f(x^{(i_1)}, t_{in}, \overline{P}^{(m-2)})\, \Delta_{in} = F^{(1)}n(x^{(i_1)}, \overline{P}^{(m-2)})$$

defines an infinite sequence of functions in $x^{(i_1)}$ each of which is bounded (numerically not greater than $K|b-a|$), is $R$-integrable in $x^{(i_1)}$ on $I$ and has a limit function, i. e.,

$$(3) \qquad \lim_{n\to\infty} F^{(1)n}(x^{(i_1)}, \overline{P}^{(m-2)}) = \int_a^b f(x^{(i_1)}, x^{(i_2)}, \overline{P}^{(m-2)})\, dx^{(i_2)},$$

for each fixed $x^{(i_1)}$ "almost everywhere" on $I$.

We now apply to the sequence (2) a generalization of a theorem due to Lebesgue * that if a bounded sequence of summable functions converges "almost everywhere" to a limit function, the limit function is summable and the integral of the limit is the limit of the integral. Furthermore, if a

* *Leçons sur l'intégration*, Gauthier-Villars, Paris, 1904, p. 111 and p. 114.

bounded function is $R$-integrable it is summable and its $R$-integral is equal to its $L$-integral.* Hence from (2) and (3) we have

$$(4) \qquad \int_a^b dx^{(i_2)} \int_a^b f(x^{(i_1)}, x^{(i_2)}, \overline{P}^{(m-2)})\, dx^{(i_1)} = L \int_a^b dx^{(i_1)} \int_a^b f(x^{(i_1)}, x^{(i_2)}, \overline{P}^{(m-2)})\, dx^{(i_2)}.$$

Hence $F^{(1)}(x^{(i_2)}, \overline{P}^{(m-2)})$ is integrable in $x^{(i_2)}$ on $I$, for $\overline{P}^{(m-2)i_1 i_2}$ not in $C_0$. This conclusion is clearly valid for any two integers $i_1$ and $i_2$ which are distinct from each other and belong to the set $1, 2, \cdots, m$.

Now all the original hypotheses as to integrability and boundedness are fulfilled for the functions $F^{(1)}(\overline{P}^{(m-1)i_1})$ since the numerical value of each is not greater than $K \mid b - a \mid$. Repeating the foregoing process therefore, we establish conclusion 1 for $j = 1$, $j = 2, \cdots, j = m - 1$ for a particular $m$. To show that it holds for all positive integral values of $m$ we follow the usual method of induction. Clearly the conclusion is valid for $m = 2$. Suppose it is not true for all positive integral values of $m$. Then there is a first one for which it is not true. Let this one be $m = r + 1$. Then the conclusion is valid for $m = r$, i. e.,

$$F^{(j)}(x^{(k)}, \overline{P}^{(r-j-1)}) = \int_a^b \cdots \int_a^b f(x^{(k)}, \overline{P}^{(r-j-1)}, P^{(j)})\, dP^{(j)}$$

is an integrable function of $x^{(k)}$ for every fixed $\overline{P}^{(r-j-1)}$ not in $C_0$. Consider now $f(P^{(r+1)}) = f(P^{(r)}, \overline{x}^{(r+1)})$ where $x^{(r+1)}$ is any fixed point of $I$ not in $C_0$. The above argument embodied in equations (1), (2), (3), (4) may now be applied to $f(P^{(r+1)})$ and we obtain that

$$F^{(j)}(x^{(k)}, \overline{P}^{(r-j)})$$

is an integrable function of $x^{(k)}$ for every $P^{(r-j)}$ not in $C_0$. Hence the conclusion 1 holds for all positive integral values of $m$.

For $j = m$, we obtain conclusion 2 viz., that all the $m$-fold iterated integrals

$$\int_a^b dt^{(i_1)} \int_a^b dt^{(i_2)} \cdots \int_a^b f(P^{(m)})\, dt^{(i_m)}$$

exist.

To show that all the $m!$ $m$-fold iterated integrals are equal, it is sufficient to show that any two orders of integration which are consecutive may be

* Lebesgue, *l. c.*, p. 112.

permuted. By making all possible changes of this kind we obtain conclusion 3. If we interchange $x^{(i_1)}$ and $x^{(i_2)}$ in the reasoning of equations (1) to (4), we obtain the fact that both of the integrals of (4) may be written as $R$-integrals. Hence the order of two consecutive simple integrations may be inverted from which conclusion 3 follows.

This result might have been obtained also from an extension of Gillespie's theorem to the effect that the iterated integrals are equal whenever they exist.

## SUMMATION THEOREM FOR ITERATED INTEGRALS.

THEOREM IV.

*Hypothesis.* 1. $f(P^{(m)})$ is a function satisfying the hypotheses of Theorem III, i. e., the $m!$ $m$-fold iterated integrals of $f(P^{(m)})$ over $S^{(m)}$ exist.

2. $H_n(P^{(m)})$ is a horizontal function of index $n^m$ in $S^{(m)}$.

3. $H_n(P^{(m)}$ is bounded for all values of $n$ and all points $P^{(m)}$ in $S^{(m)}$, i. e., $| H_n(P^{(m)}) | < \bar{K}$, where $\bar{K}$ is a constant independent of $n$ and $P^{(m)}$.

4. If $\bar{x}_{n_k}{}^{(k)}$ is a set of values of $x^{(k)}$ on $I$, such that $\lim_{n_k \to \infty} \bar{x}_{n_k}{}^{(k)} = \bar{x}^{(k)}$, then

$$\lim_{n_m \to \infty} \left( \lim_{n_{m-1} \to \infty} \cdots \left( \lim_{n_1 \to \infty} H(\bar{x}_{n_1}{}^{(1)}, \bar{x}_{n_2}{}^{(2)}, \cdots, \bar{x}_{n_m}{}^{(m)}) \right) \cdots \right) = f(\bar{P}^{(m)})$$

"almost everywhere" * where $\bar{P}^{(m)}$ is a fixed point of $S^{(m)}$.

*Conclusion.*

$$\lim_{n \to \infty} \sum_{i_1 = i_2 = \cdots i_m = 1}^{n} H_{i_1 i_2 \cdots i_m ; n} \quad \Delta_n{}^m = \int_a^b \cdots \int_a^b f(P^{(m)}) dP^{(m)}.$$

Proof. For $m = 1$, the present theorem reduces to Theorem II. Suppose the theorem is not true for all positive integral values of $m$. Then there is a first one for which it is not true. Call it $m = r + 1$. Then the theorem is true for $m = r$, i. e.,

$$\lim_{n \to \infty} \sum_{i_1 = i_2 \cdots = i_r = 1}^{n} H_{i_1 i_2 \cdots i_r ; n} \; \Delta_n{}^r = \int_a^b \cdots \int_a^b f(P^{(r)}) dP^{(r)}.$$

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* If $M_0$ is a null set on $I$, the limit is to exist for all values of $x^{(k)}$ except at most for those in $M_0$.

Consider $H_n(P^{(r)}, x^{(r+1)})$. For any fixed $x^{(r+1)}$ on $I$ except at most for a null set, this reduces to the case $m = r$. Now

$$\lim_{n\to\infty} \sum_{i_1=i_2=\cdots=i_{r+1}=1}^{n} H_{i_1 i_2 \cdots i_{r+1};\, n} \quad \Delta_n{}^{r+1}$$

$$= \lim_{n\to\infty} \sum_{i_{r+1}}^{n} \Delta_n \left( \sum_{i_1=\cdots i_r=1}^{n} H_{i_1 i_2 \cdots i_{r+1};\, n} \quad \Delta_n{}^r \right).$$

(5) Let $H_{i_{r+1};\, n} = \sum_{i_1=i_2=\cdots=i_{r+1}=1}^{n} H_{i_1 i_2 \cdots i_r;\, n} \quad \Delta_n{}^r.$

Then the numerical value of $H_{i_{r+1};\, n}$ is not greater than $K \mid b - a \mid^r$. Also

$$\lim_{n\to\infty} H_n(x^{(r+1)}) = \lim_{n_{r+1}\to\infty} \left( \lim_{n\to\infty} H_n(P^{(r)}, x_{n_{r+1}}^{(r+1)}) \right)$$

$$= \lim_{n_{r+1}\to\infty} \int_a^b \cdots \int_a^b f(P^{(r)}, x_{n_{r+1}}^{(r+1)}) dP^{(r)} = F^{(1)}(x^{(r+1)}),$$

for every fixed $x^{(r+1)}$ on $I$ except at most a null set, since $F^{(1)}(x^{(r+1)})$ is by Theorem III integrable on $I$, and therefore continuous except for a null set.

The conditions of Theorem II are now satisfied for (5). Applying Theorem II we obtain the present theorem for the case of $m = r + 1$. Hence the theorem is true for all positive integral values of $m$.

From Theorem III we obtain the following two corollaries.

COROLLARY I. Under the hypotheses of Theorem III

$$F(P^{(j)}) = \int_a^b dt^{(j+1)} \cdots \int_a^b f(P^{(m)}) dt^{(m)}$$

is a continuous function of $x^{(k)}$, $(k = 1, 2, \cdots, j)$ "almost everywhere" on $I$, for every set of values $(x^{(1)}, \cdots, x^{(k-1)}, x^{(k+1)}, \cdots, x^{(j)})$ not in $C_0$.

We have already used a special case of this corollary at one stage of the proof of Theorem IV.

COROLLARY II. If $x^{(i)}$ is not in $C_0$, $(i = 1, 2, \cdots, m)$,

$$F(P^{(m)}) = \int_a^{x^{(i_2)}} dt^{(i)} \cdots \int_a^{x^{(i_m)}} f(P^{(m)}) dt^{(i_m)}$$

has all partial cross derivatives except at points of $C_0$. These cross derivatives are independent of the order in which the derivatives are obtained.

$$F_x{}^{(i_1)} = \int_a^{x^{(i_2)}} dt^{(i_2)} \cdots \int_a^{x^{(i_m)}} f(x^{(i_1)}, t^{(i_2)}, \cdots t^{(i_m)} dt^{(m)},$$

which is continuous in each $x^{(k)}$ of $P^{(m-1)i_1}$, for each point not in $C_0$,

$$F_{x^{(i_1)}x^{(i_2)}} = \int_a^{x^{(i_2)}} dt^{(i_2)} \cdots \int_a^{x^{(i_m)}} f(x^{(i_1)}, x^{(i_2)}, t^{(i_3)}, \cdots t^{(i_m)})\, dt^{(m)},$$

which is continuous in each $x^{(k)}$ of $P^{(m-1)i_1 i_2}$ for each point not in $C_0$,

$$\vdots$$

$$F_{x^{(i_1)}x^{(i_2)} \cdots x^{(i_m)}} = f(x^{(i_1)}, \cdots, x^{(i_m)}.$$

The work of this paper may be extended to include the iterated integrals in which a higher integral than the single integral is involved at some or all of the steps. Under the assumption of integrability in $j$ $(j < m)$ independent variables for fixed values of the others except for points of $C_0$, the results will include all possible orders which involve repeated $j$-fold multiple $(j-1)$-fold multiple, $\cdots$, double or single integrals. The case $j = m$ yields precisely the existence of the $m$-fold multiple integral.

The above results also permit of certain extensions to more general types of integration.

THE UNIVERSITY OF TEXAS.

# Cubic Involutions and a $C_3$.

By H. S. White.

In the plane of the homogeneous coordinates $(x_1, x_2, x_3)$, let two cubic involutions be given by the equations

$$F(x_1, x_3) = t \cdot x_3^3, \quad f(x_2, x_3) = -t \cdot x_3^3.$$

Then will corresponding triads of rays in the two pencils intersect and generate the cubic curve

$$F(x_1, x_3) + f(x_2, x_3) = 0;$$

together with the triply counting line: $x_3 = 0$, of which we shall make no further mention. Conversely, every plane cubic curve can be generated in this way. The proof consists in finding a transformation which will throw its equation into the above form, separating the variables $x_1$ and $x_2$.

Denote the equation of any proper curve of the third order by $(ax)^3 = 0$. The problem is, to determine two points $(\alpha)$ and $(\beta)$ which may serve as vertices of a new triangle of reference, that is, as centers of the two involutions. Write these equations for the collineation:

$$\begin{aligned} x_1 &= \alpha_1 y_1 + \beta_1 y_2 + \gamma_1 y_3, \\ x_2 &= \alpha_2 y_1 + \beta_2 y_2 + \gamma_2 y_3, \\ x_3 &= \alpha_3 y_1 + \beta_3 y_2 + \gamma_3 y_3. \end{aligned}$$

In order that the desired separation may occur, it is necessary and sufficient that three terms vanish, namely those in $y_1^2 y_2$, $y_1 y_2^2$, and $y_1 y_2 y_3$ respectively. Their coefficients are

$$(a\alpha)^2(a\beta) = 0, \quad (a\alpha)\,(a\beta)^2 = 0, \quad (a\alpha)\,(a\beta)\,(a\gamma) = 0.$$

Since the third point, $(\gamma)$, is not to be collinear with $(\alpha)$ and $(\beta)$, these are equivalent to the three conditions:

$$(a\alpha)\,(a\beta)\alpha_1 = 0, \quad (a\alpha)\,(a\beta)\alpha_2 = 0, \quad (a\alpha)\,(a\beta)\alpha_3 = 0.$$

Hence we see that points $(\alpha)$ and $(\beta)$ must be conjugate points on the Hessian of the given cubic; and further, that any pair of conjugate points may be taken as vertices of the new triangle of reference, with the third point arbitrary, to reduce the equation of the cubic to the required form.

Accordingly we have on the cubic curve a "Grid" in Coble's sense (see Vol. 43 of this *Journal*, p. 12), of the special kind whose triads radiate from two points; and this Grid has a poristic character in that nine points can move along the cubic with one degree of freedom while the two centers are stationary on the Hessian, and also the two centers are free to slide on the Hessian while nine, or at least eight of the crossing points follow the fundamental cubic.

Observe also that the theorem here established gives immediately the theorem of Cayley or Cremona, that the satellite point of a tangent to the Cayleyan (Quippian) has the Hessian as its locus.

Extensions to the cubic surface are not without interest.

VASSAR COLLEGE, March, 1926.